Chapter 18

Algebra for Physics

If you aren't comfortable with algebra now, this chapter will help you make quick progress toward mastery and confidence. If you do know algebra, use it as a review, and to help you organize your knowledge into a logical system.

You can use these sections in any order: 18.1 (how to make equations; overcoming fear of "the unknown"), 18.2 (Basic Algebra: equation-solving tools, arithmetic for algebra, inequalities), 18.3 (fractions and percentages), 18.4 (overall equation-solving strategy), 18.5 (math bloopers!), 18.6 (exponents, scientific notation and logarithms), 18.7 (quadratic equations), 18.8 (solving multi-unknown "simultaneous equations"), 18.9 (ratio logic), 18.10 (how to find a maximum or minimum), and 18.11 (using a calculator for physics).

Earlier, Chapter 1 covered these topics: line geometry, vectors and trigonometry, metric & S.I. units, and conversion factors.

Section 10.93 discusses volume & surface-area formulas for spheres, cylinders,...

Section 20.4 discusses attitudes (toward math and science, exams and life).

Optional Sections: 18.91 & 18.92 (problems & answers), =={these aren't in order} Significant Figures, Trigonometry for non-90° triangles, Trig Formulas for Angle-addition and Angle-Multiplication, Advanced Multiple-Unknown Strategy, and a Summary of SI Units.

18.1 How to Make Equations

For most physics problems, the equation you need already exists as a statement-of-relationship like "$\Delta x = v\,\Delta t$" (from Chapter 2) or "$F = ma$" (from Chapter 3). Just choose an equation and use it. Strategies for choosing equations are explored in Sections 2.4, 4.12 and 5.6b.

You can also MAKE an equation, by using the concept of a *balance-scale equation:* an "=" sign declares that an equation's left side is exactly the same (in "kind of stuff" and the amount of it) as its right side, like equal weights on a balance scale.

Balance-scale logic is used to solve this problem: At a store, Swiss cheese costs 4.00 dollars/pound, and cheddar cheese is $3.00/pound. If you buy 3 pounds (of mixed Swiss & cheddar) and pay $10.20, how much money did you spend on Swiss cheese?

To solve this problem, use the "**Total = Sum of Parts**" principle to make an equation. { If "x" represents the pounds of Swiss, what is (x)(4.00)? Hint: What are the units of x? of 4.00? }

$$\begin{array}{lcccc} \text{TOTAL COST} & = & & \text{TOTAL COST} & \\ \text{TOTAL COST} & = & \text{Cost of Swiss} & + & \text{Cost of cheddar} \\ 10.20 \text{ dollars} & = & (x)(4.00) & + & (3-x)(3.00) \\ 10.20 & = & 4.00x & + & 9.00 - 3.00x \\ 1.20 & = & x & & \end{array}$$

This balance scale equation doesn't become useful until the right-side total cost is expressed in a new way, as the sum of two partial costs. These partial costs can then be described in algebraic terms: (x pounds)(4.00 dollars/pound) is the Swiss-cheese dollars; notice how the units cancel. Similarly, (3–x)(3) is the cheddar dollars.

10.20, x(4) and (3–x)(3) are *equation terms.* As required by balance-scale logic, each term is the same "kind of thing" [money-cost] in the same units [dollars].

overcoming fear of "the unknown"

Concepts like "2" () and "5" () are easy and comfortable. But "x" doesn't seem as real; you can't see it or touch it, and you can't count to "x" on your fingers.

What does "unknown" really mean? Is it the deep mystery implied by its name?

In the Cheese Problem, from the time we decided to let "x" represent the number of pounds of Swiss cheese, x was 1.20, even though we didn't know it until the end when we solved the equation. The columns below show algebra from two viewpoints: limited knowledge (with an "unknown x") and knowing the reality (with 1.20).

What the algebra looks like:	**What is really happening:**
$10.20 = (x)(4) + (3-x)(3)$	$10.20 = (1.20)(4) + (3-1.20)(3)$
$10.20 = 4x + 9 - 3x$	$10.20 = 4(1.20) + 9 - 3(1.20)$
$1.20 = 1x$	$1.20 = 1(1.20)$

Notice that algebra manipulations of "x" are really being done to "1.20". Do you see that there is no algebraic difference between x and 1.20? **If you can do an algebra operation with a number (like 1.20), you can also do it with a letter-unknown (like x).**

Compare these questions and answer them:

A) If Joe and Sue together have $100, and Joe has $40, how much does Sue have?

B) If Joe and Sue together have $100, and Joe has $Y, how much does Sue have?

Problem A is easy. Subtract, 100 – 40, to get $60. Problem B is also easy. Just do the analogous subtraction, with "Y" taking the place of "40". Sue has "100 – Y".

Here is another way to solve Problem A: just think "to make $100 total, Sue must have $60", without doing a conscious subtraction. This intuitive method is fine with numbers, but to get the analogous Y-solution you must ask "<u>How</u> did I get 60?" and answer "by subtracting 100 – 40".

When you create an equation, if you know the numerical value of something, use it. If not, define it symbolically in terms of a letter like "x". In algebra, "unknown" doesn't mean vague, strange, difficult or dangerous; an *unknown* is a real number whose value is temporarily unknown.

18.2 Basic Algebra

Equation-Solving Tools

Some equations (like $3 + 4 = 7$) are always true. Others (like $3 + x = 7$) are true only if x has a specific numerical value. The usual goal of algebra is to "solve" for this value, by using one or more of these equation-solving tools.

1) **SUBSTITUTE**: Replace any part of the equation with an equal counterpart.
2) **DO THE SAME THING TO BOTH SIDES OF THE EQUATION**: $+ - \times \div$, and more.
3) **DO STANDARD ARITHMETIC OPERATIONS**: add, multiply, factor, simplify,...
4) **MULTIPLY ANY PART OF THE EQUATION BY "1"**: clever, useful ways to do this are discussed in Sections 1.7 (Conversion Factors) and 18.3 (Fractions).

Notice how each operation preserves the equation's "balance": if an equation's left and right sides were equal, they will still be equal after you use any of these tools.

Examples of these four operations are discussed in Sections 18.2 and 18.3. Section 18.4 combines them into a logical, dependable equation-solving strategy.

Substitution

If you know that $y = 7$, you can remove "y" and replace it with "7".
It is good problem-solving logic to **USE VERTICAL SUBSTITUTION**.
Don't substitute horizontally; by definition, an equation has 2 (and only 2) sides.

Use vertical substitution,	not horizontal substitution.
$3 + x = y$	$3 + x = y = 7$
$\Downarrow$	or
$3 + x = 7$	$y = 3 + x = 7$

(Occasionally, to save space, this book runs equations sideways to show a continuing "operation" on one side of an equation. For example, $x = 4(2+1) = 4(3) = 12$.)

"Same Thing" Techniques

If an equation's left & right sides are equal, and you do the same thing to both sides, the left and right sides will still be equal. Notice how the tools below are used to make progress toward "isolating +1x by itself".

ADDITION	SUBTRACTION	MULTIPLICATION	DIVISION
$x - 7 = 5$	$3 = x + 5$	$\frac{2}{3}x = 14$	$3x = 15$
add 7	subtract 5	multiply by $\frac{3}{2}$	divide by 3
$+7 + x - 7 = 5 + 7$	$-5 + 3 = x + 5 - 5$	$(\frac{3}{2})\frac{2}{3}x = 14(\frac{3}{2})$	$\frac{3x}{3} = \frac{15}{3}$
$x = 12$	$-2 = x$	$x = 21$	$x = 5$

It is sometimes easier to isolate x on the right side, to get "$5 = x$" instead of "$x = 5$".

Each equation-part that is separated by a + or – sign is called a *term*. For example, in "7 + x", 7 and x are terms. When two (or more) things are multiplied together, each multiplier is a *factor*. In "7x", 7 & x are factors.

The operations of algebra affect terms and factors differently.

In the "shortcuts" shown below, the 7-term remains a term after it changes sides, but its ± sign is changed. When the 7-factor changes sides (dividing both sides by 7 makes 7 disappear from the left side), it moves from on-the-top to on-the-bottom.

7 is a "term"	7 is a "factor"
$7 + x = 21$	$7x = 21$
$x = 21 - 7$	$x = \frac{21}{7}$

{ Similar shortcuts are possible for most other algebra operations. }

The operations below also "do the same thing" to both sides. You can raise each side to the same power, take the logarithm of each side, or use each side as an exponent to which the same number (for example, 10 or e) is raised: $10^{\text{left side}} = 10^{\text{right side}}$.

{ Power-exponents and logarithms are explained clearly in Section 18.6. }

SQUARE

$$\begin{aligned} \sqrt{x} &= 3 \\ (\sqrt{x})^2 &= (3)^2 \\ x &= 9 \end{aligned}$$

SQUARE ROOT

$$\begin{aligned} x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \\ x &= \pm 3 \end{aligned}$$

+3 and −3 are both solutions

CUBE ROOT

$$\begin{aligned} x^3 &= 64 \\ \sqrt[3]{x^3} &= \sqrt[3]{64} \\ x &= +8 \end{aligned}$$

−8 isn't a solution because $(-8)^3 \neq 64$

log

$$\begin{aligned} 10^{x-2} &= 3 \\ \log(10^{x-2}) &= \log(3) \\ x-2 &= .477 \\ x &= 2.477 \end{aligned}$$

10ˣ

$$\begin{aligned} \log(x-2) &= 3 \\ 10^{\log(x-2)} &= 10^3 \\ x-2 &= 1000 \\ x &= 1002 \end{aligned}$$

ln

$$\begin{aligned} e^{x-2} &= 3 \\ \ln(e^{x-2}) &= \ln(3) \\ x-2 &= 1.099 \\ x &= 3.099 \end{aligned}$$

eˣ

$$\begin{aligned} \ln(x-2) &= 3 \\ e^{\ln(x-2)} &= e^3 \\ x-2 &= 20.08 \\ x &= 22.08 \end{aligned}$$

Two (or more) equations can be added, subtracted, multiplied or divided:

$$\left.\begin{aligned} \text{If} \quad 2x - 7 &= 16, \\ \text{and} \quad \underline{3x + 7} &= \underline{19}, \\ \text{then} \quad 5x \quad &= 35. \end{aligned}\right\}$$

Why is this is a "same thing operation"? Adding "3x + 7" to the left side of "2x – 7 = 16" is the same as adding "19" to its right side, because "3x + 7" = "19".

ADDING EQUATIONS can help you solve Simultaneous Equations (Section 18.8).
DIVIDING EQUATIONS is the foundation for one form of Ratio Logic (Section 18.9).

Arithmetic for Algebra

ALGEBRAIC ADDITION & SUBTRACTION

Combine each "kind of thing" independently: all x^2's, all x's, all y's, all numbers.

$$z = 2 - 3x - x^2 + 5 - 7y - 3 + 2y + 3x^2$$
$$z = (-x^2 + 3x^2) + (-3x) + (-7y + 2y) + (+2 + 5 - 3)$$
$$z = \quad +2x^2 \qquad -3x \qquad -5y \qquad +4$$

ALGEBRAIC MULTIPLICATION AND DIVISION

As with addition, each "kind of thing" [numbers, u, x, y, z]
is multiplied or divided independently.

$$a = \frac{20\,u^4\,x^3}{y^3} \bullet \frac{3\,x\,y^2}{4\,u\,z}$$
$$a = \frac{15\,u^3\,x^4}{y\,z}$$

Addition, multiplication and division of fractions is summarized in Section 18.3.

If an "even number" of negative numbers are multiplied or divided, the result is +. But if the negatives are an "odd number" (1, 3, 5,...), the result is –. For example,

$(+2)(+3)(-4) = -24$	$\frac{(+4)(-6)}{(-2)(+3)} = +4$	$\frac{(+4)(-6)}{(-2)(-3)} = -4$	$-(-2)(-3)(-4) = +24$
1 negative (odd)	2 negatives (even)	3 negatives (odd)	4 negatives (even)

In the "COMBINED" MULTIPLICATION-AND-ADDITION below,
each term inside the first ( ) is multiplied times each term inside the second ( ):

$$y = (-4x + 2)(3x^2 + 7x - 2)$$
$$y = -12x^3 - 28x^2 + 8x + 6x^2 + 14x - 4$$

{ Do you see what causes each of the 6 terms? }

FACTORING is the reverse of combined multiplication-and-addition. For each group [numbers, x, y, z,...], ask "What can be moved outside the parentheses?".

$$a = 12\,x^3\,y^2\,z + 36\,x^2\,z^2$$

after transformation by
the process of "factoring"

$$\downarrow$$

$$a = \underline{12\,x^2\,z}\,(1\,x^2\,y^2 + 3\,z)$$

What number can be moved? { $\underline{12}$ }

What x-power can be moved? { $\underline{x^2}$ }

What y-power can be moved? {none}

What z-power can be moved? { $\underline{z}$ }

A check: If you multiply the two factors, do you get the original?

An exponent-power, logarithm or trigonometry-function acts on only the thing, either a number or letter or ( ), that is directly beside it. For example,

EXPONENTS		LOGARITHMS		TRIG
$(2-5)^2$	$3(5)^2$	$3\log 100 + 900$	$\log(100 + 900)^2$	$3\sin 60° + 30$
$(-3)^2$	$3(25)$	$3\ (2)\ + 900$	$\log\{(1000)^2\}$	$3\ (.5)\ + 30$
$+9$	75	906	6	31.5

An exception: 10^{x-2} always means $10^{(x-2)}$.

Also, be careful to interpret log a b and sin a b as intended. The usual meanings are log(ab) and sin(ab), but they can also be interpreted as (log a) b or (sin a) b.

ORDER OF OPERATION

Do multiplication-and-division before addition-and-subtraction: $400 + 3 \times 2 = 400 + 6$.
But operations within parentheses are always done first: $(400 + 3)2 = (403)2 = 806$.

If anything is enclosed by **ABSOLUTE VALUE** signs, | |, its ± sign will be +.
For example, $|-3| = +3$, $|+3| = +3$, and $-|-3| = -(+3) = -3$.

Inequality "Equations"

Sometimes an "equation" doesn't specify the exact numerical value of an unknown; instead, it states that the unknown's value must fall within a certain limited range.

For example, if $x \le 3$ (which means x is "less than or equal to" 3), then x could be any number that is less than 3 (like –11, 0, +2.99), or x can be equal to 3, but x cannot be greater than 3 (it couldn't be +3.01, +4, +10,...).
$x \le 3$ and $3 \ge x$ have the same meaning.

$<$ means *is less than*
$\le$ means *is less than or equal to*
$\ll$ means *is much less than*

$>$ means *is greater than*
$\ge$ means *is greater than or equal to*
$\gg$ means *is much greater than*

$\approx$ means *is approximately equal to*
$=$ means *is exactly equal to*
$\ne$ means *is not equal to*
$\equiv$ means *is defined to be equal to*
$\propto$ means *is proportional to*

An "inequality" is solved as if it was an "equality", with one exception:
if you divide both sides of an inequality-equation by a negative number, you must "reverse" the inequality sign. For example,

Two correct solution options.		Incorrect!
$7 - x \le 5$	$7 - x \le 5$	$7 - x \le 5$
$-x \le 5 - 7$	$7 - 5 \le +x$	$-x \le 5 - 7$
$-x \le -2$	$+2 \le x$	$-x \le -2$
$x \ge +2$		$x \le +2$

18.3 Fractions

To **MULTIPLY** two or more fractions, multiply their tops & bottoms independently.

The 30x/60xy fraction below is "reduced" by dividing its top & bottom by 30x. This is equivalent to multiplying by 1, so it doesn't change the value of the fraction.

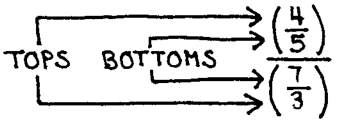
$$\frac{2}{5x}\cdot\frac{3x}{4y}\cdot\frac{5}{3} \Rightarrow \frac{2\cdot 3x\cdot 5}{5x\cdot 4y\cdot 3} \Rightarrow \frac{30x}{60xy} \Rightarrow \frac{30x}{60xy}\,\frac{\div 30x}{\div 30x} \Rightarrow \frac{1}{2y}$$

In this shortcut method, the 5's, x's and 3's are *canceled* (as shown by the /'s):

$$\frac{2}{\cancel{5}\cancel{x}}\cdot\frac{\cancel{3}\cancel{x}}{4\,y}\cdot\frac{\cancel{5}}{\cancel{3}} \Rightarrow \frac{\cancel{2}^{1}}{{}_{2}\cancel{4}y} \Rightarrow \frac{1}{2\,y}$$

DIVISION. In the example below, the fraction-top and fraction-bottom are multiplied by 3/7. { Multiplying something by 1 doesn't change it, and 3/7 ÷ 3/7 = 1. } The overall result is invert-and-multiply; invert the bottom & multiply times the top. At the right is a good "visual shortcut"; the extreme highs & lows (top-of-the-top & bottom-of-the-bottom) finish on top, while the two middle positions are on the bottom.

$$\frac{\left(\frac{4}{5}\right)}{\left(\frac{7}{3}\right)} \Rightarrow \frac{\left(\frac{4}{5}\right)}{\left(\frac{7}{3}\right)}\cdot\frac{\left(\frac{3}{7}\right)}{\left(\frac{3}{7}\right)} \Rightarrow \frac{\left(\frac{4}{5}\right)\left(\frac{3}{7}\right)}{1}$$



$$\frac{\left(\frac{4}{5}\right)}{\left(\frac{7}{3}\right)} = \frac{4\cdot 3}{5\cdot 7}$$

Using this shortcut, do you see why every fraction below has the same meaning? Can you find other fractions that have this meaning? { For some answers, see Problem 18-#. }

$$\frac{AD}{BC} = \frac{\left(\frac{AD}{BC}\right)}{1} = \frac{A}{\left(\frac{BC}{D}\right)} = \frac{\left(\frac{A}{B}\right)}{\left(\frac{C}{D}\right)} = \frac{\left(\frac{1}{B}\right)}{\left(\frac{C}{AD}\right)} = \frac{1}{\left(\frac{BC}{AD}\right)} = \frac{\left(\frac{D}{BC}\right)}{\left(\frac{1}{A}\right)}$$

If an equation's left & right sides are both fractions, it can be solved by same-thing multiplying or by "shortcuts": *cross-multiplying* and *moving variables diagonally.*

SAME-THING MULTIPLYING	CROSS MULTIPLYING	DIAGONAL MOVEMENT
$\frac{10}{A} = \frac{5}{4}$	$\frac{10}{A} = \frac{5}{4}$	$\frac{10}{A} = \frac{5}{4}$
$4A\,\frac{10}{A} = \frac{5}{4}\,4A$	$10\cdot 4 = 5A$	$\frac{10\cdot 4}{5} = \frac{A}{1}$
$4\cdot 10 = 5A$	$8 = A$	$8 = A$

Do you see the two diagonals, shown by ><, that are cross-multiplied?

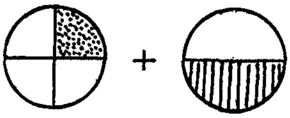
Do you see the 3 diagonal movements that are used to isolate x on top and by itself?

Do you see why diagonal movements leave the cross-multiplying (and thus the equation) unchanged?

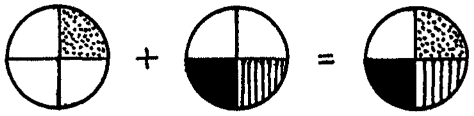
COMBINING FRACTIONS: Addition & Subtraction

Fractions can only be combined if their bottoms are the same.
As shown below, $\frac{1}{4}$ pie and $\frac{1}{2}$ pie are different sizes and they cannot be "added": $\frac{1}{4} + \frac{1}{2} =$?

If the second pie is cut into 4 pieces, two $\frac{1}{4}$-size pieces (which equal one $\frac{1}{2}$-size piece) can be added to the first pie's one $\frac{1}{4}$-size piece, to get three $\frac{1}{4}$-size pieces: $\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$.



$$\frac{1}{4} \quad + \quad \frac{1}{2} \quad = \quad ?$$



$$\frac{1}{4} \quad + \quad \frac{2}{4} \quad = \quad \frac{3}{4}$$

If the fraction-bottoms (the *denominators*) aren't the same, they can be made equal. Just multiply each fraction by "1", in whatever form is needed to get a *common denominator* (which is 20 & 12 in the middle & right examples, respectively).
{ How can you find a "common denominator"? This is discussed in Problem 18-#. }

$$\frac{3}{8} + \frac{1}{8} \qquad\qquad \frac{1}{4} - \frac{3}{5} \qquad\qquad -\frac{1}{4} + \frac{2}{3} + \frac{5}{6}$$

$$\frac{3+1}{8} \qquad\qquad \frac{1}{4}\left(\frac{5}{5}\right) - \frac{3}{5}\left(\frac{4}{4}\right) \qquad\qquad -\frac{1}{4}\left(\frac{3}{3}\right) + \frac{2}{3}\left(\frac{4}{4}\right) + \frac{5}{6}\left(\frac{2}{2}\right)$$

$$\frac{4}{8} \qquad\qquad \frac{5}{20} - \frac{12}{20} \qquad\qquad -\frac{3}{12} + \frac{8}{12} + \frac{10}{12}$$

$$\frac{1}{2} \qquad\qquad -\frac{7}{20} \qquad\qquad \frac{15}{12}$$

A fraction larger than 1 can be written as an integer-and-fraction combination:
$\frac{15}{12} = 1\frac{3}{12} = 1\frac{1}{4} = 1.25$. Dividing 15 by 12 (try it on your calculator) also gives 1.25.

Subtraction can be done by finding a *common denominator* (several examples were given earlier) or, as shown below, by changing both fractions to decimals:

$$\frac{5x}{4} - \frac{x}{2} \quad \Rightarrow \quad \frac{5}{4}x - \frac{1}{2}x \quad \Rightarrow \quad 1.25x - .50x \quad \Rightarrow \quad +.75x$$

Fractions and Percentages

If a train contains 82 dogs, 152 ducks, 28 cows, 59 cabbages, 14 rocks and 47 cats, what fraction of the animals are dogs? What decimal fraction? What percent?

$$\text{fraction of dogs} \equiv \frac{\text{number of dog-animals}}{\text{total number of animals}} = \frac{\text{PART}}{\text{WHOLE}}$$

$\frac{\text{82 dogs}}{82 + 152 + 28 + 47}$ ---(do the calculation)→ .265 ---(multiply by 100)→ 26.5 %

⇑ ⇑ ⇑

FRACTION **DECIMAL FRACTION** **PER-CENT**

The latin word "centrum" means "100". English words like century (100 years) and cent (1/100 dollar) are derived from it. Per centage (or percent, %) means, literally, per 100. To find the % of dogs, ask "If the ratio of "dogs/total animals" remained the same as in the actual situation, how many dogs would there be per 100 total animals ?

There are 3 variables in the % formula:

$$\text{PERCENT} = \frac{\text{PART}}{\text{WHOLE}} \times 100$$

If you know 2 of these 3 (PERCENT, PART, WHOLE), you can find the third.
For example, if you know that 82 dogs is 26.5% of the total animals, substitution gives $26.5 = \frac{82}{\text{TOTAL ANIMALS}} \times 100$, which can be solved for "total animals = 309".

If a problem asks you to find "percent change", use this formula:

$$\text{PERCENT CHANGE} = \frac{\text{FINAL VALUE} - \text{INITIAL VALUE}}{\text{INITIAL VALUE}} \times 100$$

18.4 Overall Equation-Solving Strategy

The following techniques can be used to solve equations.
(Same-Thing and Fraction Operations were discussed in Sections 18.2 and 18.3.)

Do the same thing to both sides: + − x ÷ , x^2 $\sqrt{\ }$ $\sqrt[3]{\ }$ ..., 10^x e^x log ln.
Fraction Operations: combine (+ −), x ÷ , cross-multiply & diagonal-moves.
Special Techniques: Quadratic (18.7), Simultaneous (18.8), Ratio Logic (18.9).

In addition, you must know how to MAKE PROGRESS toward a solution.
Most equations can be solved using a logical, systematic procedure. For example,

BASIC FORMAT

$$1.6x^2 + 7 = 2.0x^2 - 3$$
$$10 = .4x^2$$
$$25 = x^2$$
$$\sqrt{25} = \sqrt{x^2}$$
$$\pm 5 = x$$

FRACTION FORMAT

$$\frac{180}{5(x+2)^2} = \frac{4}{1}$$
$$180 = 20(x+2)^2$$
$$9 = (x+2)^2$$

Use QUADRATIC TOOLS from Section 19.7

$$\Downarrow$$

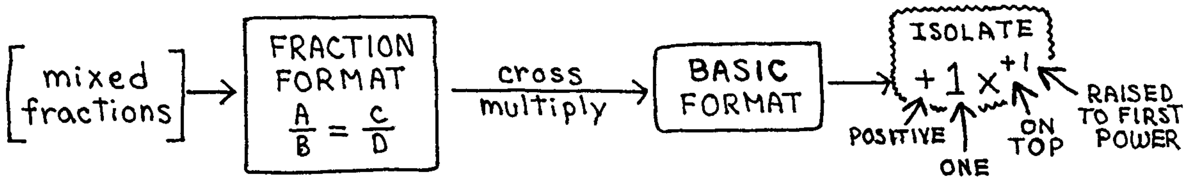
$$x = +1, \text{ or } x = -5$$

combine to get FRACTION FORMAT

$$\frac{1}{2} = \frac{3}{8x} - \frac{4}{5x}$$
$$\frac{1}{2} = \frac{3}{8x}\left(\frac{5}{5}\right) - \frac{4}{5x}\left(\frac{8}{8}\right)$$
$$\frac{1}{2} = \frac{15}{40x} - \frac{32}{40x}$$
$$\frac{1}{2} = \frac{-17}{40x}$$
$$\frac{x}{1} = \frac{2(-17)}{40}$$
$$x = -.85$$

If equation is **BASIC FORMAT**, 1) Isolate x's on one side, non-x's on other side.
2) Divide to get +1 in front of x. (The left example can also be solved by isolating x's on the left, $-.4x^2 = -10$, then divide both sides by $-.4$ to get $+1\,x^2 = +25$.)
3) If necessary, do what is needed [x^2, $\sqrt{\ }$, $\sqrt[3]{\ }$, quadratic tools (Section 18.7)] to get x^{+1}.

If necessary, combine fractions (as in right example) to get **FRACTION FORMAT**, then cross-multiply (middle example) or move variables diagonally (right example) to get BASIC FORMAT, then isolate $+1\,x^{+1}$. Here is an overall strategy summary:



Other solution methods are discussed later: exponents and logarithms {18.6}, quadratic equations {18.7}, multiple-unknown systems {18.8}, ratio logic {18.9}. And if you can't get a solution using algebra, try "successive approximations".

Successive Approximations: A *solution* is the x-value that makes an equation's left side equal to its right side. To find a solution by ***guess-and-check*** *successive approximations* (trial-and-error?), just substitute different values of x until you find one that makes the equation's left & right sides equal. For example, there is no easy way to solve "$x^3 + x = 25$" with regular algebra, but it can be solved:

x	=	2	3	2.7	2.8	2.9	2.81	2.82
$x^3 + x$	=	10	30	22.38	24.752	27.289	24.998	25.246
error?		low	high	low	low	high	low	high

For each x-guess (2, 3, 2.7,...), $x^3 + x$ is calculated (to give 10, 30, 22.38,... in the second row) and compared with the right-side value of "25". Then use "successive approximations" logic to decide whether the next x-guess should be higher or lower.

The 2-guess (which gives an "$x^3 + x$" of 10) is too low, while the 3-guess (it gives 30) is too high. Do you see why this tells you that x is between 2 and 3? 30 is closer to 25 than is 10, so x is probably closer to 3 than to 2, and we'll make the next guess be 2.7. Further guess-and-check logic shows that the value of x is between 2.8 and 2.9 (do you see why?) and then between 2.81 and 2.82. { Is x closer to 2.81 or to 2.82? } If we wanted a more precise solution for x, we could continue this guess-and-check process.

In "real life" physics, engineering or biology, when correction factors (for air resistance,...) are included, equations are more complex and often cannot be solved using regular algebra. But a computer or calculator can usually be programmed to find a solution using "guess and check" successive approximation.

18.5 Math Bloopers

In this collection of students' favorite blunders, many mistakes are **almost logical**. A common error is to disregard the "if" requirement of "If.... , then...." reasoning, and use an algebra technique when the situation doesn't permit its use.

It is possible to be too creative with "algebra analogies", as these examples show: 1) To multiply fractions, we multiply the tops together & the bottoms together; to add fractions, can we add the tops together & add the bottoms together? 2) Fractions can be "combined" if their bottoms match; does this also work if their tops match?

With letters (like x, y & z) it's hard to tell if these questionable analogies are true. But with numbers (like 3, 4 & 5) it's easy to do a *number check*. To test whether an algebra technique is valid, compare the number-value of the original expression with its value after the technique [shown by → below] has been used. For example,

TESTING ANALOGY #1

$\frac{4}{5}\cdot\frac{3}{4} \rightarrow \frac{4\cdot 3}{5\cdot 4}$

$(.80)(.75) \overset{?}{=} \frac{12}{20}$

$.60 \overset{?}{=} .60$

YES

→ TECHNIQUE IS CORRECT

$\frac{4}{5}+\frac{3}{4} \rightarrow \frac{4+3}{5+4}$

$.80+.75 \overset{?}{=} \frac{7}{9}$

$1.55 \overset{?}{=} .78$

NO

→ TECHNIQUE IS INCORRECT

TESTING ANALOGY #2

$\frac{3}{5}+\frac{4}{5} \rightarrow \frac{3+4}{5}$

$.6+.8 \overset{?}{=} \frac{7}{5}$

$1.4 \overset{?}{=} 1.4$

YES

→ TECHNIQUE IS CORRECT

$\frac{5}{3}+\frac{5}{4} \rightarrow \frac{5}{3+4}$

$1.67+1.25 \overset{?}{=} \frac{5}{7}$

$2.92 \overset{?}{=} .71$

NO

→ TECHNIQUE IS INCORRECT

This kind of number check can prove that an operation is incorrect, but it cannot prove that an operation is correct. The reason for this is discussed in "Scientific Method", Section 20.7.

When you test the truth of an algebra operation with numbers, don't use 0 or 1.

Know (and use!) the "If..., then..." conditions for every algebra technique. An example: If the fraction bottoms (not tops) match, then they can be combined. If you understand the logical reason for an "If..." condition (like the pie-fraction explanation in Section 18.3), you will be able to remember it more easily.

At the left below: You can CROSS-MULTIPLY if (and only if) an equation's entire left side is one fraction, and its entire right side is also one fraction. At the right: You can FLIP both sides if (and only if) each side is one fraction.

WRONG

$\frac{x}{4}+\frac{x}{5}=\frac{9}{5}$

ERROR ↓

$\frac{x}{4}+5x \neq 45$

CORRECT

$\frac{x}{4}+\frac{x}{5}=\frac{9}{5}$

$\frac{x}{4}\left(\frac{5}{5}\right)+\frac{x}{5}\left(\frac{4}{4}\right)=\frac{9}{5}$

$\frac{5x+4x}{20}=\frac{9}{5}$

CORRECT

$\frac{1}{x}=\frac{1}{4}$

FLIP ↓

$\frac{x}{1}=\frac{4}{1}$

WRONG

$\frac{1}{x}=\frac{1}{4}+\frac{1}{5}$

ERROR ↓

$\frac{x}{1}\neq\frac{4}{1}+\frac{5}{1}$

Below: the entire "x + 3" factor, not just the first term "x", is cross-multiplied by 2. Notice how the use of ( )'s encourages correct algebra-thinking and technique. At right: for DIAGONAL MOVEMENTS, entire factors must be moved as a unit.

WRONG

$\frac{2}{3}=\frac{8}{x+3}$

$2x+3 \neq 24$

CORRECT

$\frac{2}{3}=\frac{8}{(x+3)}$

$2x+6=24$

CORRECT

$\frac{2}{3}=\frac{8}{(x+3)}$

$\frac{x+3}{1}=\frac{3\cdot 8}{2}$

As shown in Section 18.3, parts of fractions can be "canceled" if they're being multiplied (as on the left below) but not if they're being added (as on the right).

$$\frac{25\,(3x)}{25\,(y)}=\frac{3x}{y} \qquad\qquad \frac{25+3x}{25+y}\neq\frac{3x}{y}$$

At the left below: the "–" multiplies everything inside the "x – 4" factor.
At the right: 2 is used once (to multiply the whole first factor), but not twice.

WRONG	CORRECT	WRONG	CORRECT
$x = -(x-4)$	$x = -(x-4)$	$x = 2(x+y)(3x+4y)$	$x = 2(x+y)(3x+4y)$
$x \neq -x-4$	$x = -x+4$	$x \neq (2x+2y)(6x+8y)$	$x = (2x+2y)(3x+4y)$

ORDER-OF-OPERATION: $-10^2 = -100$, but $(-10)^2 = (-10)(-10) = +100$.

INCONSISTENT ALGEBRA SYMBOLISM? When a letter is next to something, it always means "multiply": 4a and a4 mean (4)(a). But things are different with numbers: $3\frac{1}{2}$ means $3 + \frac{1}{2}$, and 32 just means "thirty two". // And 10^{-1} means 1/10, but "$\sin^{-1} x$" is not "1/ sin x". { The meaning of $\sin^{-1} x$ is explained in Section 1.3.}

Don't write 1/2x, which is usually interpreted as 1/(2x), if what you mean is $\frac{1}{2}x$.
Either write fractions with —, not /, or use parentheses to clarify your meaning.
Here is another example: 1/4+x means $\frac{1}{4} + x$, while 1/(4+x) means $\frac{1}{4+x}$.

This fraction, $\dfrac{\dfrac{3}{5}}{9}$, is ambiguous. Parentheses clarify the meaning as $\dfrac{\left(\dfrac{3}{5}\right)}{9}$ or $\dfrac{3}{\left(\dfrac{5}{9}\right)}$.

Exponents and Logarithms

EXPONENT RULES, as summarized in Section 18.6, are for × and ÷, not for + and –.
Notice the requirement: <u>if</u> you multiply or divide (not add or subtract), <u>then</u> you can use the rule.

for these situations, RULES CAN BE USED	for these situations, RULES CAN'T BE USED
$10^3 10^2 = 10^{3+2}$ $\frac{10^3}{10^2} = 10^{3-2}$	$10^2 + 10^3 = ?$
$16^{1/2} 25^{1/2} = (16 \cdot 25)^{1/2}$ $\sqrt{16}\sqrt{25} = \sqrt{16 \cdot 25}$ $\frac{\sqrt{16}}{\sqrt{25}} = \sqrt{\frac{16}{25}}$	$\sqrt{16} + \sqrt{25} \neq \sqrt{16+25}$
$\begin{array}{ccc} (10^a 10^b)^c & = & 10^{ac} 10^{bc} \\ \downarrow & & \uparrow \\ (10^{a+b})^c & \rightarrow & 10^{ac+bc} \end{array}$	$(10^a + 10^b)^c \neq 10^{ac} + 10^{bc}$

To use the *matched base* and *matched exponent* rules, there must be a "match": $10^3 10^2 = 10^{3+2}$ [the 10-bases match], and $10^3 5^3 = (10 \cdot 5)^3$ [the 3-exponents match], but there is no rule [because there is no match] for $10^3 5^2$.

There are logarithm rules for multiplication & division: for example, $\log(10^3 10^2) = \log(10^3) + \log(10^2)$. But there are no log-rules for "$\log(10^3 + 10^2)$" or "$(\log 10^3)(\log 10^2)$".

Learn techniques "forward" and "backward". Are you as comfortable with $\log(10^3) + \log(10^2) \Rightarrow \log(10^3 10^2)$ as with $\log(10^3 10^2) \Rightarrow \log(10^3) + \log(10^2)$?

Rules for logs (and for trigonometry) aren't the same as rules for regular numbers: $a(bc) = abc$, but $[a \log(bc)] \neq \log(abc)$; instead, $[a \log(bc)] = \log(bc)^a$. Similarly, $\cos(ab) \neq (\cos a)(\cos b)$, and $\cos(a + b) \neq \cos a + \cos b$.

18.6 Exponents, Scientific Notation, Logarithms

Exponents

Here is the meaning of a base-and-exponent combination:
multiply <u>this number</u> times itself <u>this number</u> of times.

$$4^3 = 4 \cdot 4 \cdot 4$$

BASE EXPONENT

The + sign in 10^{+3} means that 10^3 goes "on the top":

$$10^{+3} = \frac{10^3}{1} = 10 \cdot 10 \cdot 10$$

The – sign in 10^{-3} means that 10^3 goes "on the bottom".

$$10^{-3} = \frac{1}{10^3} = \frac{1}{10 \cdot 10 \cdot 10}$$

If you move a base-exponent combination from bottom to top (or vice versa), you must change the exponent's ± sign:

$$\frac{1}{10^{-4}} = \frac{10^{+4}}{1} \quad \text{and} \quad \frac{1}{10^{+4}} = \frac{10^{-4}}{1}$$

Fractional exponents (like $10^{5/2}$) are discussed later in this section.
Often, parentheses are not used in exponents: 10^{x+3} means $10^{(x+3)}$.

The rules below are true for <u>any positive base</u>* [10, e (whose numerical value is 2.718...), or variable-letters like "B"], and for <u>any exponent</u> [either + or –, integer or fraction or decimal, number or letter]. * In some cases, negative bases can't be used.

MATCHED-BASE MULTIPLICATION & DIVISION		POWER	MATCHED-EXPONENT MULTIPLICATION & DIVISION	
$B^x B^y = B^{x+y}$	$\frac{B^x}{B^y} = B^{x-y}$	$(B^x)^y = B^{xy}$	$A^x B^x = (AB)^x$	$\frac{A^x}{B^x} = \left(\frac{A}{B}\right)^x$
$10^2 \cdot 10^3$ $(10 \cdot 10)(10 \cdot 10 \cdot 10)$ 10^5 10^{2+3}	$\frac{e^3}{e^2}$ $\frac{e \cdot e \cdot e}{e \cdot e}$ e^{+1} e^{3-2}	$(8^2)^3$ $(8^2)(8^2)(8^2)$ $(8 \cdot 8)(8 \cdot 8)(8 \cdot 8)$ 8^6 $8^{2 \cdot 3}$	$10^2\ 4^2$ $(10 \cdot 10)(4 \cdot 4)$ $(10 \cdot 4)(10 \cdot 4)$ $(10 \cdot 4)^2$	$10^3 / 4^3$ $\frac{10 \cdot 10 \cdot 10}{4 \cdot 4 \cdot 4}$ $\frac{10}{4} \cdot \frac{10}{4} \cdot \frac{10}{4}$ $\left(\frac{10}{4}\right)^3$

The examples above should convince you that each "rule" is reasonable.

To show that $\underline{B^0 = 1}$, use the "division rule": $\underline{1} = B^2/B^2 = B^{+2-2} = \underline{B^0}$.

FRACTIONAL EXPONENTS: To find the $\sqrt{\ }$ of 25, ask "What number, if raised to the second power, will give 25?". Two answers to this question are "+5" and "−5". Another answer is "$\sqrt{25} = 25^{1/2}$", because $(25^{1/2})(25^{1/2}) = (25^{1/2})^2 = 25$. Similarly, $\sqrt[3]{100} = 100^{1/3}$, $\sqrt[4]{B} = B^{1/4}$, and so on.

$B^{-1/2} = \frac{1}{\sqrt{B}}$; the "−" (not the fraction-exponent of $\frac{1}{2}$) puts $\sqrt{B}$ on the fraction-bottom.

$10^{5/2} = 10^{2.5}$ and, by using the power rule, $10^{5/2} = (10^{1/2})^5 = (10^5)^{1/2}$. You can use a calculator {Section 18.11} to find that $10^{2.5} = 316.2$, $(10^{1/2})^5 = (3.162)^5 = 316.2$, and $(10^5)^{1/2} = (100000)^{1/2} = 316.2$.

{ $10^{2.0} = 100$, and $10^{3.0} = 1000$, so wouldn't you expect $10^{2.5}$ to be between 100 & 1000? And it is! }

Scientific Notation

Scientific notation (underlined below) is useful for very large & very small numbers.

$2340 = 2.34 \times 1000 = \underline{2.34 \times 10^{+3}}$ $\quad\quad .00234 = 2.34 \times .001 = \underline{2.34 \times 10^{-3}}$

In the tables below, notice that the exponent shows the number of places the decimal point moves (from where it is in 1.0), not the number of 0's.

1000.	=	1000.	=	10^3	=	$10 \cdot 10 \cdot 10$
100.	=	100.	=	10^2	=	$10 \cdot 10$
10.	=	10.	=	10^1	=	10
1.	=	1.	=	10^0	=	1
.1	=	.1	=	10^{-1}	=	1/10
.01	=	.01	=	10^{-2}	=	$1/(10 \cdot 10)$
.001	=	.001	=	10^{-3}	=	$1/(10 \cdot 10 \cdot 10)$

A number (like 100.0, which is 10^2) is made smaller by a multiplying factor of 1/10 if its decimal point is moved 1 place to the left (to get 10.00) or if its 10^x exponent is decreased by 1 (to get 10^1). It is made larger by a factor of 10 if its decimal point is moved 1 place to the right (to get 1000.) or its exponent is increased by 1 (to get 10^3).

In the third example below, the "483" part of "483×10^3" is made smaller by a factor of 1/100 when its decimal point is moved 2 places to get 4.83. To compensate, the "10^3" part increases by a factor of 100 when its exponent changes by 2 to get 10^5. Because (1/100)(100) = 1, these two changes cancel each other, and the size of the original number (483×10^3) and final number (4.83×10^5) is the same.

The other four examples show similar "counter-balancing changes". Be careful when you use this logic with negative exponents: 10^{-5} is larger than 10^{-7}.

$$\begin{array}{cc} 2340 & \times\ 1 \\ \boxed{\times\frac{1}{1000}} & \boxed{\times 1000} \\ \downarrow & \downarrow \\ 2.340 & \times\ 10^3 \end{array} \qquad \begin{array}{cc} .00234 & \times\ 1 \\ \boxed{\times 1000} & \boxed{\times\frac{1}{1000}} \\ \downarrow & \downarrow \\ 2.34 & \times\ 10^{-3} \end{array} \qquad \begin{array}{cc} 483 & \times\ 10^3 \\ \boxed{\times\frac{1}{100}} & \boxed{\times 100} \\ \downarrow & \downarrow \\ 4.83 & \times\ 10^5 \end{array} \qquad \begin{array}{cc} 483 & \times\ 10^{-7} \\ \boxed{\times\frac{1}{100}} & \boxed{\times 100} \\ \downarrow & \downarrow \\ 4.83 & \times\ 10^{-5} \end{array} \qquad \begin{array}{cc} .483 & \times\ 10^{-7} \\ \boxed{\times 10} & \boxed{\times\frac{1}{10}} \\ \downarrow & \downarrow \\ 4.83 & \times\ 10^{-8} \end{array}$$

Here is a good way to think intuitively about these conversions. 483×10^3 is larger than 10^3 (or 4.83×10^3); to correctly show this, when the decimal is moved 2 places, the 10^x exponent must be made larger by 2 (to get 10^5) and not smaller by 2 (to get 10^1). On the other hand, $.483 \times 10^{-7}$ is smaller than 10^3 so the 10^x part of the number will become smaller when you convert to 4.83×10^{-8}.

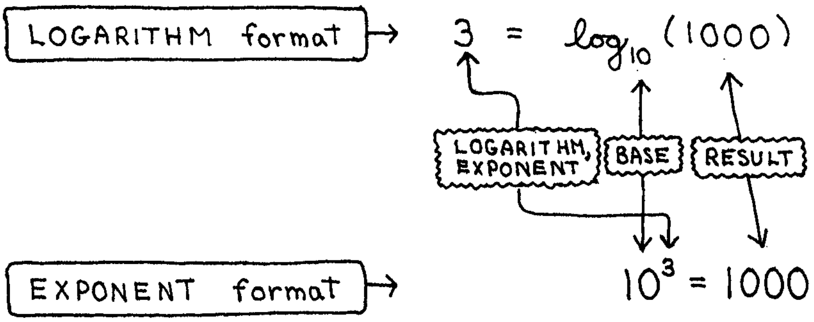
As explained in Section 18.11, most calculators have a special button that makes it easy to use scientific notation.

Logarithms

LOGARITHM QUESTIONS : When you ask "What is the logarithm of 1000?" or "log 1000 = ?", it really means "What number do you have to raise 10 to, in order to get 1000?", or "$10^? = 1000$". The answer to each of these four "logarithm questions" is the same: $\log(1000) = +3$, because $10^{+3} = 1000$.

The exponent (which is +3) is the log, even though 1000 is next to the word "log" in "log 1000 = ?".

Notice the roles that log-exponent and base and result play in the two formats below:



BASES: In science, two logarithm-bases are commonly used: 10 and e. e is a number that, like π, occurs in many math/science formulas; e = 2.71828...

The log-base (10, e, 2, ...) can be shown with a subscript ($\log_{10}$, $\log_e$, $\log_2$, ...), but the two standard logarithm-bases (10 & e) are usually just abbreviated as log & ln. "log 1000" is called the "log of 1000", while "ln 1000" is called the "natural log of 1000".

"natural-log questions" are identical to "log questions", but 10's are replaced by e's: These ln-questions { What is the natural log of 1000?, ln 1000 = ?, What power do you have to raise e to in order to get 1000?, $e^? = 1000$ } are answered "6.9078", because $e^{6.9078} = 1000$.

e (which is **2.71828...**) is smaller than **10**, so e must be raised to a higher exponent in order to get the same result: the exponent to which e must be raised (which is the ln) is always larger than the exponent to which 10 must be raised (which is the log) by a factor of 2.3026...

This fact can be stated in exponent format, $e^{2.3026\,y} = 10^y$, or in logarithm format:

$$\ln(x) = 2.3026 \log(x)$$

For example, ln 1000 {which is 6.9078} is 2.3026 times as large as log 1000 {which is 3.0000}.

There is a **Logarithm Rule** that corresponds to each exponent rule:

MULTIPLICATION	DIVISION	POWERS	zeros
$\ln JK = \ln J + \ln K$	$\ln \frac{J}{K} = \ln J - \ln K$	$\ln J^y = y \ln J$	$\ln 1 = 0$
$e^x e^y = e^{x+y}$	$e^x / e^y = e^{x-y}$	$(e^x)^y = e^{xy}$	$1 = e^0$

To get the analogous rules for base-10 logarithms, just replace "ln" with "log".
By careful study, you can discover the relationship between the log-rules and exponent-rules.
Hint: In the rules above, $J = e^x$ and $K = e^y$. { Or check the derivations in Problem 18-#.}

HOW TO TRANSFORM LOGARITHMIC & EXPONENTIAL EQUATIONS:

"What is the log of 10^3 ?" asks "What power do you have to raise 10 to, in order to get 10^3 ?" Think about it, and you'll find the easy answer to these questions.

To answer a different question, "What is $10^{\log 1000}$?", think about what a log means: log(1000) is defined as the number such that, if you raise 10 to it, you'll get 1000. So what happens when you raise 10 to that particular power? You get 1000, of course!

Here is a summary of these operations, for both log & 10^x, ln & e^x:

$\log 10^x = x$ $\quad$ $10^{\log x} = x$ $\quad$ $\ln e^x = x$ $\quad$ $e^{\ln x} = x$

A MEMORY TRICK — When you see 10 & log (or e & ln) next to each other, remember these rules. In the combinations above (shown by ◌'s), the 10^x and log (or e^x and ln) "undo" each other.
These rules work for any exponent. For example, $\log 10^{3.4x - 2yz} = 3.4x - 2yz$.

<u>To transform an exponent equation</u> (with 10^x or e^x) into a logarithm equation, take the logarithm (log or ln) of both sides of the equation.
<u>To transform a logarithm equation</u> (with log x or ln x) back into an exponent equation, use this logic: if leftside = rightside, then $10^{\text{leftside}} = 10^{\text{rightside}}$.
For example, here are two transformations that use these "same thing" techniques:

EXPONENT EQUATION → $\left[\; y = 5e^{x-2}\right.$

$$\frac{y}{5} = e^{x-2}$$

$$\ln\frac{y}{5} = \ln(e^{x-2})$$

$$\ln\frac{y}{5} = x-2$$

LOGARITHM EQUATION → $\left[\; \ln\frac{y}{5} + 2 = x\right.$

$\left.\ln\frac{y}{5} + 2 = x \;\right]$ ← LOGARITHM EQUATION

$$\ln\frac{y}{5} = x - 2$$

$$e^{\ln\frac{y}{5}} = e^{x-2}$$

$$\frac{y}{5} = e^{x-2}$$

$\left.y = 5e^{x-2}\;\right]$ ← EXPONENT EQUATION

How do you know which type of equation to use for "substitute-and-solve"?
It is easy to decide if you use this logical (and thus easily remembered) principle:
If you want to solve for a certain variable, <u>use the equation where it is "free"</u>, where it is not being used as an exponent (in 10^x or e^x) or a result (in log x or ln x).

For example, if you know that x = 4.5, it is easy to solve "$y = 5e^{x-2}$" for "y = 60.9".
If you know that y = 60.9, it is easy to solve for x (because it is "free") in "$\ln\frac{y}{5} + 2 = x$".

A log equation can be transformed into a ln equation (or vice versa), and a 10^x equation can be transformed into an e^x equation (or vice versa); just use "$\ln x = 2.303 \log x$" or "$e^{2.303\,y} = 10^y$" for substitution.

For example, "$A = \ln x$" is the same as "$A = 2.303 \log x$", and $B = 10^x$ is $B = e^{2.303\,x}$.

18.7 Quadratic Equations: a detour, some tricks, and The Formula.

Consider these 3 kinds of terms: x^2, x and numbers. The examples below show that when an equation contains only two kinds of terms, it is easy to solve.

x and number

$$\begin{aligned} 6x - 7 &= 11 \\ 6x &= 18 \\ x &= 3 \end{aligned}$$

x^2 and number

$$\begin{aligned} 2x^2 - 4 &= 14 \\ 2x^2 &= 18 \\ x^2 &= 9 \end{aligned}$$
$$(x = +3,\ x = -3)$$

x^2 and x

$$\begin{aligned} 2x^2 - 6x &= 0 \\ x(2x - 6) &= 0 \end{aligned}$$
$$x = 0 \qquad 2x - 6 = 0$$
$$x = 3$$

An equation with all 3 kinds of terms is called "quadratic". An example is "$4x^2 - 16x + 16 = 100$", which is equivalent to "$(2x - 4)^2 = 100$". A *quadratic equation* requires special solution methods; two of the most useful are shown below.

the $\sqrt{\ }$ trick

$$\begin{aligned} (2x-4)^2 &= 100 \\ \sqrt{(2x-4)^2} &= \sqrt{100} \\ (2x-4) &= \pm 10 \end{aligned}$$

$\oplus$
$$\begin{aligned} 2x - 4 &= +10 \\ 2x &= 14 \\ x &= 7 \end{aligned}$$

$\ominus$
$$\begin{aligned} 2x - 4 &= -10 \\ 2x &= -6 \\ x &= -3 \end{aligned}$$

the QUADRATIC FORMULA

$$\begin{aligned} (2x-4)(2x-4) &= 100 \\ 4x^2 - 16x + 16 &= 100 \\ 4x^2 - 16x - 84 &= 0 \\ \underbrace{+1}_{a=+1}x^2 \ \underbrace{-4}_{b=-4}x \ \underbrace{-21}_{c=-21} &= 0 \end{aligned}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$\underbrace{x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(+1)(+21)}}{2(+1)}}$$

$\oplus$
$$x = \frac{+4 + 10}{2} \qquad x = +7$$

$\ominus$
$$x = \frac{+4 - 10}{2} \qquad x = -3$$

To find **a**, **b** and **c** for the Quadratic Formula, put the Quadratic Equation into "$\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = 0$" format.

The $\sqrt{\ }$-trick produces + and – roots, so there are two possible solutions: +7 and –3. Similarly, the Q-formula's $\sqrt{\ }$ has ± roots, which give two solutions: +7 and –3.

EVALUATION: Think about the "physical meaning" of each solution and decide whether either, neither or both are physically meaningful. For example, in Problem 2-A of Section 2.6, a rocket accelerates upward at +16.0 m/s^2 for 5.00 seconds, then shuts off its engines. What is its final just-before-crashing velocity, and liftoff-to-impact time? In Section 2.6, we found two solutions for final velocity: +101.6 m/s (the + means the rocket is traveling upward; this is wrong) and –101.6 m/s (this downward v is correct). There are two answers for shutoff-to-impact time: +18.5 s and –4.4 s. Time runs "forward", so the correct Δt is +18.5. { The meaning of the discarded solutions, $v_f = +101.6$ m/s and $\Delta t = -4.4$ s, is discussed in Problem 18-#.}

Notice that the correct solution choice can be either – (as with v_f) or + (as with Δt).

Q-FORMULA CALCULATIONS: The diagram below shows an efficient calculator method. A) Punch " 4 x 1 x 21 = ". Look at the number of negative numbers that are multiplied (there are two: –4, –21) and decide whether it should be +84 (yes) or –84 (no; but if there were an odd number of –'s, you would change +84 to –84 with the "+/–" button). B) Punch " + 4 x^2 = √‾ ", and put the result (which is 10) into the calculator's memory or write it next to the √‾ as shown below. C) To get the two solutions, punch " clear 4 + memoryrecall(or 10) = ÷ 2 = " to get +7, and then " clear 4 – memoryrecall(or 10) = ÷ 2 = " to get –3.

$$\frac{\overbrace{-(-4)\pm}^{C,D}\ \overbrace{{}^{10}\sqrt{(-4)^2}}^{B}\overbrace{\ -4(+1)(-21)}^{A}}{2}$$

Here are five ways to solve quadratic equations: √‾-trick, Quadratic Formula, factoring, quadratic detour, and successive approximations.

When an equation is in a form that allows use of the <u>√‾-trick</u>, this is the easiest solution method. The <u>Quadratic Formula</u> can be used for any situation; if the number inside the √‾ is –, either there is no "real number" solution, or the negative number is a result of a mistake during substitution or algebra.

Sometimes <u>factoring</u> can be used. Change "$x^2 - 4x - 21 = 0$" into "$(x - 7)(x + 3) = 0$", and then use logic. One or more of the multipliers, $(x - 7)$ and $(x + 3)$, must be zero; either "$x - 7 = 0$" to give $x = +7$, or "$x + 3 = 0$" to give $x = -3$. {This method often works in math class, because teachers give equations that can be "factored", but it is seldom used for science or engineering because for most combinations of a, b & c [like $x^2 - 4x - 20 = 0$ or $x^2 - 3x - 21 = 0$ or...], the quadratic equation can't be split into factors.}

In the 5-variable/5-equation "tvvax system" of Section 2.4, when you reach the 3-of-5 goal there are two missing variables. If the equation you choose by using 2.4's "1-out strategy" is quadratic, you have two choices. You can solve it. Or you can avoid it by using the <u>quadratic detour</u> that is described in Section 2.6 — solve for the "unwanted variable" first, then solve for the variable you really want.

<u>Guess-and-check "succesive approximations"</u> may be useful, especially if you can program a computer (or calculator) to do the repetitive calculations. This method is discussed in Section 18.4. { If you've studied acid-base reactions in a chemistry course, you may have used succesive approximations strategy to solve the equation "$K_a = x^2 / (C_i - x)$". }

{ For *cubic equations* like $x^3 + 2x^2 - 4 = 0$ or $x^3 - 5x - 3 = 0$ or $x^3 + 2x^2 - 5x - 3 = 0$, succesive approximations is usually the only solution method. }

18.8 Solving for Two (or more) Unknowns, by using Simultaneous Equations

If two equations (like $3x + y = 11$ and $5x - 3y = 11$) are *true at the same time*, they are called *simultaneous equations.*

These equations can be solved using LEAPFROG SUBSTITUTION. Study the solution below, one step at a time. Notice the back-and-forth leapfrog pattern. Action begins on the left (Step 1, **solve**), moves to the right (Steps 2 & 3, **substitute & solve**), then back to the left (Steps 4 & 5, **substitute & solve**).

$$\left.\begin{aligned}3x + y &= 11\\ y &= 11 - 3x\end{aligned}\right\} ①$$

$$\Downarrow ④$$

$$\left.\begin{aligned}y &= 11 - 3(+2)\\ y &= 11 - 6\\ y &= +5\end{aligned}\right\} ⑤$$

$$5x - 3y = -5$$

$$\Downarrow ②$$

$$\left.\begin{aligned}5x - 3(11-3x) &= -5\\ 5x - 33 + 9x &= -5\\ 14x &= +28\\ x &= +2\end{aligned}\right\} ③$$

Notice how Steps 1-5 alternate: solve, substitute, solve, substitute, solve. **Each time you solve**, whether it is a "partial" solution with letters (like $y = 11 - 3x$) or a "total" solution for a definite number (like $x = +2$ or $y = +5$), **use the solution-result**; substitute it into the other equation.

Optional Check: After solving the left equation for y, substitute $x = +2$ and $y = +5$ into the right equation; the left & right sides will be equal if x and y are correct.

There are 4 leapfrog orders. For the first step you can solve for y on the left (this was done above) or y on the right, x on the left or x on the right. If one solution-order gives difficult algebra, you can either solve it anyway or try a different order.

(If you want, try the last three leapfrog-orders; they will give the same solutions of $x = 2$ and $y = 5$.)

Another solution method is ADDING EQUATIONS.
In the left example, do you see what was done to "$3x + y = 11$", and why?

$$\begin{aligned}5x - 3y &= -5\\ \underline{9x + 3y} &\underline{= +33}\\ 14x &= +28\\ x &= +2\end{aligned}$$

$$\begin{aligned}+A &= 5.0a\\ -A + B &= 2.0a\\ -B + C &= 4.7a\\ \underline{+29.4 - C} &\underline{= 3.0a}\\ +29.4 &= 14.7a\end{aligned}$$

In the right example, which is used in Problem 3-B of Section 3.3, adding equations quickly eliminates A & B & C, to give an easy 1-unknown equation.

At the left, good strategy causes cancelling. Why is "$3x + y = 11$" multiplied by 3? So it will, when added to "$5x - 3y = -5$", cancel the y's to give a 1-unknown equation.

Occasionally, simultaneous equations can be solved by DIVIDING EQUATIONS. For example, if you know that $4x^2y = 72$ and $2xy = 12$, divide $4x^2y = 72$ by $2xy = 12$ to produce $4x^2y/2xy = 72/12$, which gives $2x = 6$. Then substitute $x = 3$ into either of the original equations and solve for $y = 2$. (These equations can also be solved by substitution.)

Which solution method is best? SUBSTITUTION is logical, easy to use, and can be used for a wide variety of problems; I suggest using it as your "main method". Then learn to recognize situations where ADDING or DIVIDING EQUATIONS will be easier, and you can take advantage of the best features of all three methods.

Solving a system with 3 or more equations usually requires a lot of algebra; Section 18.# discusses a good way to cope with it using leapfrog substitution. An equation-adding solution can be organized with *matrices;* this optional technique, which is explained in math books, makes it easier for you (or a computer) to do the many calculations that a many-equation system requires.

Simultaneous equations can be solved if the number of *independent equations* (as defined below) **is equal to or greater than the number of unknowns**. For example, 3 unknowns in 3 equations can be solved, but 3 unknowns in 2 equations cannot.

Exception: a 2-unknown equation can sometimes be solved for one (but not both) of the unknowns. For example, when "$3mx - 2m = 4m$" is divided by m, it becomes the easy-to-solve "$3x - 2 = 4$".

A definition: *independent equations* cannot be derived from each other. $3x-2=4$ and $3mx-2m=4m$ and $6x-4=8$ and $6x=12$ are only 1 independent equation, because all 4 equations can be derived from each other by "doing the same thing to both sides". (If there is a non-independent equation, it will usually warn you at some point during the solution by producing a trivial result like "0 = 0".)

18.9 Ratio Logic

If your money-amount increases from \$200 to \$1000, by how much does it change?

There are two ways to answer this question. You can subtract, to find that money increases by \$1000 – \$200 = \$800. Or you can divide, to find that money increases by a *multiplying factor* of \$1000/\$200 = 5.

Ratio logic uses multiplying factors to predict how a change in one variable will affect other variables. This section will help you improve your ratio logic, a skill that is (in my opinion) the most practical mathematical tool you can have.

To answer Problems 1 & 2 below, use "$y = v_i t + \frac{1}{2} a t^2$" from Section 2.4, where y is distance traveled, v_i is initial speed, t is time of travel, and a is acceleration.

Problem 1: Car L's constant velocity is 3 times that of Car N.
If N goes 100 miles in a certain time, how far will L travel in the same time?
If N travels 120 miles in 6 hours, how long will it take L to drive this same distance?

Problem 2: Car K's constant acceleration is 4 times that of Car M;
the cars race from rest, and leave the starting line at the same time.
When K has gone 100 meters, how far has M traveled?
If M travels 100 meters in a certain time, how far will it go if it drives 3 times as long?
If M reaches the finish line in 16 seconds, how long does it take K to finish the race?
If K travels 100 meters in "T" seconds, how far will M travel in "3T" seconds?

Solution 1

Velocity is constant, so $a = 0$, $y = v_i\, t + \frac{1}{2}(0)\, t^2$, and the equation is "$y = v\ t$".

If you have a good intuitive sense about ratios, you'll think "L goes 3 times as fast, so (in the same time) it will go 3 times as far, which is 300 miles", and "L is 3 times as fast, so (to go the same distance) it will take 1/3 as much time, which is 2 hours".

This intuitive logic is a great way to solve problems, and I suggest that you use it. But you should also have the "equation-based tool" that is explained below, for ratio problems that are more complicated.

The first question involves **two kinds of information**: **absolute** (N goes 100 miles) and **relative** (L's speed is 3 times as fast, and both cars travel for the same time).

At the left below, relative L-info (the size of L-variables compared with N-variables) is listed above "y = v t". Because L's v is 3 times as large as M's v, "x 3" is written above the v. Because L-time is the same as N-time, "x 1" is written above the t. These two multiplying factors (x 3 and x 1) make the equation's right side 3 times larger, so to keep the equation balanced its left side must also be multiplied by a factor of 3. This conclusion is shown by a "x 3" inside the circle. (It will be easier for you understand what is happening when you do it. Write "y = v t" on a separate page. Then re-read this paragraph and, as the multiplying factors are described, write them above your own equation.)

To answer the first question, you must combine relative and absolute knowledge. L's y is 3 times as large as N's y (which is 100): L's y = 3 x 100 = 300 miles.

The second question is answered at the right below. The multiplying factors for y and v are "x1" and "x3". To counterbalance v's "x3" factor, t's factor must be "x1/3". Then use common sense; L's t is 1/3 of N's t (which is 6), so L's t = 1/3 x 6 = 2 hours.

The equal-signs on the L-information lines below state that "left-side multiplying factors = right-side multiplying factors": (x3) = (x3)(x1) and (x1) = (x3)(x1/3).

$$\begin{array}{ll} \text{RELATIVE INFORMATION for L, compared with N} \ \} & \boxed{\times 3} = \times 3 \quad \times 1 \\ & y \ = v \quad t \\ \text{ABSOLUTE INFO for N} \ \} & 100 \\ & \downarrow \\ & 3 \times 100 = y_L \end{array}$$

$$\begin{array}{ll} \text{RELATIVE for L} \ \} & \times 1 = \times 3 \ \boxed{\times \frac{1}{3}} \\ & y = v \quad t \\ \text{ABSOLUTE for N} \ \} & 120 \qquad 6 \\ & \qquad\qquad \downarrow \\ & t_L = \frac{1}{3} \times 6 \end{array}$$

v and y are **directly proportional**: when v increases (↑) by a factor of 3, so does y. In the three equations below, y and v are either on opposite sides of the equation [with both on top], or on the same side [with one on top, the other on the bottom].

v and t are **inversely proportional**: when v ↑ by a factor of 3, t ↓ {it is multiplied by a factor of 1/3}. This occurs because v and t are either on the same side [with both on top], or on opposite sides [with one on top, the other on the bottom].

Inverse proportionality is illustrated by numbers that multiply to give a constant, like 4 x 6 = 24. If the multiplication result is to stay constant at "24" and one of the multipliers is doubled, the other must be cut in half. These two changes (multiplying by 2 and by 1/2) cancel each other, so the numbers still multiply to give 24. For example, 4(x 2) x 6(x 1/2) = 8 x 3 = 24. Similarly, 4(x 1/2) x 6(x 2) = 2 x 12 = 24.

3rd question: Do you see why two t-factors of 1/2 are needed, not one factor of 1/4 ?
4th question: The two changes combine to give a multiplying factor of 9/4.

QUESTION #3

$$\text{for K}\} \times 1 = \times 4 \boxed{\times\tfrac{1}{2} \ \times\tfrac{1}{2}}$$
$$y = \tfrac{1}{2}\, a \ \ t \ \ t$$
$$\text{for M}\} \qquad 16$$
$$\downarrow$$
$$t_K = \tfrac{1}{2} \times 16 = 8s$$

QUESTION #4

$$\text{for M}\} \boxed{\times\tfrac{9}{4}} = \times\tfrac{1}{4}\,(\times 3)^2$$
$$y = \tfrac{1}{2}\, a \ t^2$$
$$\text{for K}\} \ 100$$
$$\downarrow$$
$$y_M = 100 \times (\tfrac{1}{4})(3)^2 = 225\,m$$

PROPORTIONALY: If $y = \frac{1}{2} a t^2$, y and a are proportional, y and t^2 are proportional, a and t^2 are inversely proportional. { By taking the $\sqrt{\ }$ of both equation-sides, the last two relationships become "$\sqrt{y}$ and t are proportional" and "$\sqrt{a}$ and t are inversely proportional".}

EQUATIONS-DIVIDING RATIO LOGIC

In Problem 2, a = 0 so $y = \frac{1}{2} a t^2$. For the M-car, $y_M = \frac{1}{2} a_M t_M^2$. And for the K-car, $y_K = \frac{1}{2} a_K t_K^2$. As shown below, we can divide the left & right sides of the M-equation by the left & right sides of the K-equation. Do you see why this "does the same thing" to both sides of the M-equation, and is thus an acceptable algebra operation?

To answer Question #4, we substitute for y_K, a_K, t_K and t_M, then solve for y_M.

$$\frac{y_M}{y_K} = \frac{\cancel{\tfrac{1}{2}}\ a_M\ t_M^2}{\cancel{\tfrac{1}{2}}\ a_K\ t_K^2}$$
$$\frac{y_M}{100} = \frac{a_M}{(4a_M)}\ \frac{(3T)(3T)}{(T)\ (T)}$$
$$y_M = 100\,(\tfrac{1}{4})(3^2)$$

Which equations correctly state that "K's acceleration is 4 times as large as M's"?

$4 \ a_K = a_M$	$a_K = 4 \ a_M$	$\frac{1}{4} \ a_K = a_M$
4 (large) = (small)	(large) = 4 (small)	$\frac{1}{4}$ (large) = (small)

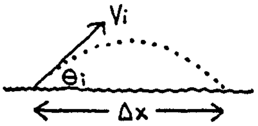
Many students think the first equation is correct, probably because they use this logic: a_K is larger, so it **deserves** to have the 4 on its side. But this is wrong. Instead, you should use this logic: a_M is smaller, so it **needs** the 4 to make it equal (as promised by the "=" sign) to the larger a_K. The last two equations are correct. The middle-equation is used to substitute for a_K above; or you can substitute for a_M using the right-equation.

18.10 How to Find a Maximum or Minimum

Here is a typical *maximization problem:* If air resistance is negligible and the shooting & landing points are the same height, what direction should a bullet be shot in order to get maximum sideways travel? {The corresponding *minimization problem* is "Find the firing-direction(s) that will produce the smallest amount of sideways travel.}

One way to find out is to *do an experiment.* Use a gun with consistent bullet-speed, shoot bullets at different angles, and observe the results.

Another option is mathematical analysis, using a formula derived in Problem 2-25: the bullet's sideways travel "Δx" is $v_i^2\,[2 \sin\theta \cos\theta]/g$, which equals $v_i^2\,[\sin 2\theta]/g$, where v_i is the bullet's initial speed, θ is the angle (above horizontal) of its initial velocity, and g is the 9.80 m/s^2 acceleration caused by gravity.



What value of θ will maximize $v_i^2(\underline{\sin 2\theta})/g$, and thus Δx? The largest possible value of $\sin 2\theta$ is 1; this occurs when $2\theta = 90°$ and $\theta = 45°$. Or you can find the "θ" that maximizes $v_i^2(\,2\,\underline{\sin\theta\ \cos\theta}\,)/g$; try different θ's, and you'll discover that this occurs when $\theta = 45°$. **{For example, (sin 44°)(cos 44°) = .49970, (sin 45°)(cos 45°) = .50000, and (sin 46°)(cos 46°) = .49970. It is easy to find "(sin 44°)(cos 44°)" with a basic calculator, but for complex formulas a computer or programable calculator is very helpful!}**

Conflicting Factors

The equation "$\Delta x = v_x\,\Delta t$" shows that Δx depends on two factors, v_x and Δt. To go far, a bullet must stay in the air (Δt) and make forward progress (v_x).

Let's look at what happens when θ is an extreme. When $\theta = 90°$, the bullet is shot vertically and Δt is maximized; but $v_x = 0$, so $\Delta x = v_x\,\Delta t = (0)\,\Delta t = 0$. When a bullet is shot horizontally with $\theta = 0$, v_x is maximized; but $\Delta t = 0$, so $\Delta x = 0$. {The "minimization problem" has two answers; $\theta = 90°$ and $\theta = 0°$ both give $\Delta x = 0$.}

For a fixed value of v_i, a decrease in θ (⤵) makes v_x increase and Δt decrease. But an increase in θ (↷) does the reverse; v_x decreases and Δt increases. Do you see that Δx is produced by two multiplying factors, v_x and Δt, that are "in conflict"? The best compromise between them occurs somewhere between the extremes of 0° & 90°, because multiplication of "MEDIUM x MEDIUM" is bigger than "LARGE x ZERO".

v_x and Δt are in conflict with respect to changes in θ (when one ↑, the other ↓). But with respect to changes in v_i, they agree. For example, v_x and Δt both ↑ when v_i ↑, thus causing a larger Δx.

Maximization and minimization problems are examples of a more general class: *optimization problems* that ask you to find the *optimum value* of a variable. The Random House Dictionary defines optimum as "the best or most favorable condition for obtaining a given result".

The optimal θ depends on the "given result" we want to achieve. To maximize Δx, $\theta = 45°$. But to get maximum Δt, $\theta = 90°$.

This section introduces a few basic optimization principles. More sophisticated principles are taught in higher physics, engineering and statistics courses.

Section 20.7 discusses the use of optimization principles in everyday life.

Problems 2-24, 3-25 and ## give valuable tips for solving optimization problems, including common sense use of the "logic of extremes", discussion of conflicting factors, and more.

The optional "max/min" calculus technique (find an equation, take the derivative, set it equal to zero and solve) is discussed in Section ~~18.1~~ 19.

18.11 Using a Calculator for Physics

The directions below work for most calculators. If you have a calculator (like an HP) that uses "reverse polish" entry, there will be some differences in the order-of-punching.

For $\frac{3 \times 10^{+2}}{5 \times 10^{-3}}$, punch "3 EXP* 2 ÷ 5 EXP +/– 3 =" to get "60000" or "6×10^{04}".

{ * TI calculators have an EE (not EXP) button. } If the calculator display is "60000", look for a button (ENG, F ↔ E, ...) that changes it to "6×10^{04}" scientific notation.

For $5.2\,(10^{-4.8})$, punch "5.2 x 4.8 +/– 10^x =" to get 8.24×10^{-5}.
For $5.2\,(e^{-4.8})$, punch "5.2 x 4.8 +/– e^x =" to get .04279.
For 5.2 log 4.8, punch "5.2 x 4.8 log =" to get 3.542.
For 5.2 ln 4.8, punch "5.2 x 4.8 ln =" to get 8.157.

{ For TI calculators, punch "INV log" and "INV ln" to get 10^x and e^x. }

As discussed later, in this section's "Order of Operation", the groupings that are underlined above are treated as if they were enclosed by parentheses. For example, "5.2 x 4.8 +/– 10^x =" is calculated as if it was "5.2 x (4.8 +/– 10^x) =".

Notice that these groupings are punched in with right-to-left order, not the left-to-right way they're written: $10^{-4.8}$ is punched "4.8 +/– 10^x", and log 4.8 is punched "4.8 log".

When to push the +/– button: To enter "5×10^{-3}", you can punch "5 EXP +/– 3" (I prefer this because 10^{-3} is spoken this way) or "5 EXP 3 +/–". But for $10^{-4.8}$ and $e^{-4.8}$, you must punch "4.8 +/–" with reversed right-to-left order; to see why, punch "+/– 4.8" and watch what happens.

With the x^y (or y^x) button, you can raise any + number to any power. For example, to find $1.2^{.71}$, punch "1.2 x^y .71 =" to get "1.138".

The $x^{1/y}$ (or $\sqrt[x]{y}$) button can be used to find any "root". To find $\sqrt[5]{100}$, punch "100 $x^{1/y}$ 5 =" to get 2.512.

Use "special buttons" for square, cube and fourth roots. For $\sqrt{100}$, $\sqrt[3]{100}$ and $\sqrt[4]{100}$, punch "100 $\sqrt{\ }$" to get 10.0, "100 $\sqrt[3]{\ }$" to get 4.64, and "100 $\sqrt{\ }$ $\sqrt{\ }$" to get 3.16.

Calculations using sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ are discussed in Section 1.3.

OOPS! Most calculators have two "clear" buttons. One clears everything, the other clears only the display (to fix a mistake). Here is a calculation of "2 x 2.5", with a fixed mistake: 2 x 25 clear-display* 2.5 = . { * Use trial-and-error to find this button. }

Section 18.7 shows an easy way to do **QUADRATIC FORMULA** calculations.

FRACTIONS

To find $\frac{1}{2} + \frac{1}{4} - \frac{1}{5}$, punch "2 $\frac{1}{x}$ + 4 $\frac{1}{x}$ − 5 $\frac{1}{x}$ =" to get ".55".

To find $\frac{5}{8} + \frac{7}{18} - \frac{2}{5}$, punch "5 ÷ 8 + (7 ÷ 18) − (2 ÷ 5) =" to get ".6139".

You must include all ('s and)'s; try leaving one out and see what happens.

For $\frac{3 \times 4}{5 \times 6}$, punch " 3 x 4 ÷ 5 ÷ 6 = " or " 3 x 4 ÷ (5 x 6) = ", to get ".4".

Let the calculator know what is on the fraction-top with "x" { x 4 }, and what is on the fraction-bottom with ÷ {÷ 5, ÷ 6}. And if you want the calculator to use ( )'s, you must "tell it" to do so; don't punch " 3 x 4 ÷ 5 x 6 " and expect the calculator to interpret it as "3 x 4 / (5 x 6)".

{ I sometimes hear students refer to 3/5 as "5 divided into 3". Don't do this; it isn't the way 3/5 is written or punched into a calculator. Instead, think of 3/5 as "3 divided by 5" or "3 over 5". }

ORDER OF OPERATION

The operation-buttons in the "upper rows" (usually everything except + – x ÷) only operate on the number that is now on the number-display. For example, if you punch "2 x 30 sin x^2 log 10^x + 1 = ", it is interpreted as "2 x (30 sin x^2 log 10^x) + 1"; multiplication by 2 doesn't occur until a lower row button (like "+") is punched.

How does your calculator interpret mixtures of x÷ with +–? To find out, punch "3 x 4 + 2 + 5 =" to get 11, and "2 + 3 x 4 + 5 =" to get 19 (or maybe 25). In each case, can you figure out what the calculator is doing? { Hint: do each calculation by hand, trying different operation-orders until you get 11 and 19 (or 25). A bonus problem: predict the calculator's answer for "2 + 3 x 4 + 3 x 4", then check to see if you're right. }

The flash-card review will begin here.