Chapter 5

Rotational Motion

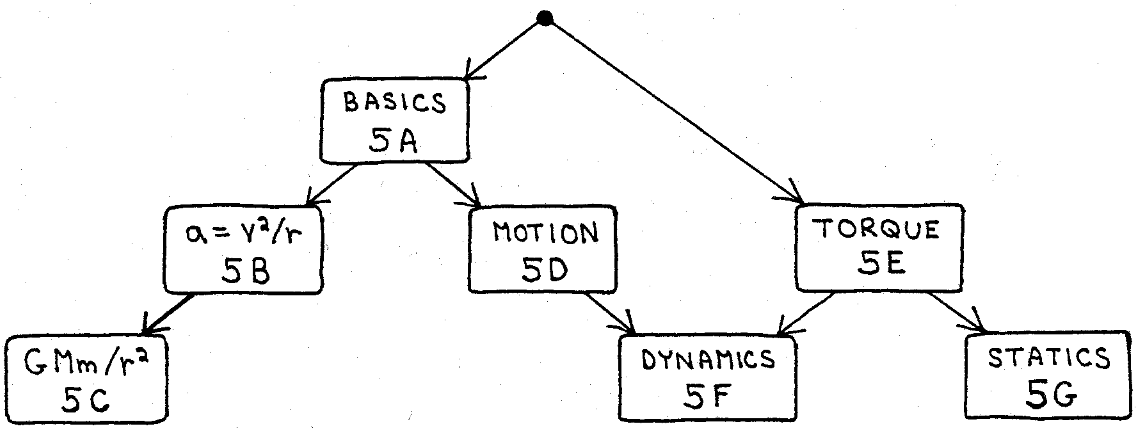
This is a special chapter. It is one of my favorites, because it organizes a wide variety of potentially confusing topics into a logical, easy-to-understand system.

Instead of the usual numbered sections (5.1, 5.2, ...), there are 7 sub-chapters: 5A to 5G. This format provides flexibility, to let you easily "match" what your class is studying — when you study a particular topic, find my corresponding sub-chapter by looking for clue-words in this summary:

5A – BASICS: ideas that are needed for the rest of Chapter 5
5B – $a = v^2/r$: radial or centripetal acceleration
5C – GMm/r^2 : gravity force, orbiting planets & satellites
5D – MOTION: analogies to tvvax equations, with variables like $\Delta\theta$, ω, α,... ; radians
5E – TORQUE: what it is, how to calculate it
5F – DYNAMICS: how torque causes angular motion to change; rotational inertia (moment of inertia), $\tau = I\alpha$, rotational kinetic energy, angular momentum
5G – EQUILIBRIUM (STATICS): using torque to analyze a non-moving situation

Bonus: The derivations in Section 5.93 show the mathematical organization of rotational motion, and its relationship with Chapters 2 to 4. It is "optional", but recommended.

The flowchart below lets you see the overall organization of Chapter 5. The arrows show possible orders-of-using. For example, you can use either 5A or 5E at the start, but for 5F you need to know 5A, 5D & 5E.

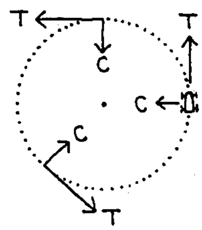


REVIEWER: Maybe these should be Sections 5.1, 5.2, ... instead of Chapters 5A, 5B, ... Parts 1, 2, 3+4 of 5D could become 5.4a, 5.4b, ... This could also be done with avoiding the split of Ch. 4 into 4A, 4B.

5A: The Basics of Circular Motion.
Centripetal & Tangential Directions

Rotation is easy to understand if you know the similarities and differences between closely related kinds of motion. Throughout Chapter 5, I'll use familiar examples to help you develop this intuitive understanding.

Imagine you are in a car, moving at constant speed around a circular racetrack. In the following view from above (a *bird's-eye view),* ⁛ is the racetrack.



To describe circular motion, it is useful to define two new axis-directions. The diagram above shows the *centripetal axis* (marked with a "C") and *tangential axis* ("T") at three different car-locations.

The centripetal axis always points toward the circle's center, along a radius-line. Because of this, it is sometimes called the *radial axis.*

The tangential axis is always perpendicular (⊥) to the centripetal axis. It points along the car's direction of motion, either forward (straight out the front windshield) or backward (straight out the rear window). { It points forward if, as is usually done, you define the + tangential direction to be in the direction the car is moving. It points backward if you define + to be in the direction opposite to motion. }

Notice that the tangential axis keeps changing its east-west/north-south direction as your car moves around the track, but the T-axis is consistent in another way: it always points in the direction the car is moving. Similarly, the E-W/N-S direction of the centripetal axis changes, but it always points toward the center of the circle.

The magnitude of the car's *instantaneous tangential velocity,* abbreviated "v_T", is the speed measured by the car's speedometer; v_T is the car's *instantaneous speed.*

THREE KINDS OF ACCELERATION

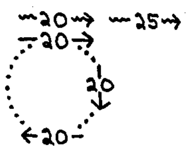
Acceleration is defined as Δ[velocity vector]/Δt. A velocity vector has magnitude and direction, so Δ[velocity vector] can occur in 3 ways: only v-magnitude changes, only v-direction changes, or both magnitude & direction change.

These 3 possibilities are shown in the bird's-eye picture below. At time "i", two cars are at the top of the circle moving eastward at 20 m/s. One car (shown by ⇝) continues eastward while the other (⟶) moves along a circular path.

A few seconds after "i", the ⇝ car is moving eastward at 25 m/s. Its v-direction is the same, but its v-magnitude has changed. This change-of-magnitude Δv/Δt is the regular *linear acceleration,* abbreviated "a", that is used in Chapters 2-4.

A few seconds after "i", the → car is moving south at 20 m/s on a circular path. Its v-magnitude has stayed the same, but its v-direction is changing. This change-of-direction Δv/Δt is called *centripetal acceleration**, abbreviated "a_c"; it is the focus of Chapter 5B. * It can also be called *radial acceleration,* "a_R". }

A little later, the car is moving west along the circle at 25 m/s. Its v-direction is still changing (so it has a_c) and it is changing speed. This change-of-magnitude Δv/Δt is called *tangential acceleration,* abbreviated "a_T"; it is studied in Chapter 5D.



25 m/s toward east (only v-magnitude changes)
20 m/s toward south (only v-direction changes)
25 m/s toward west (v-magnitude changes, while v-direction is also changing)

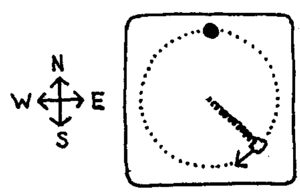
There is only one kind of acceleration: **a-vector** ≡ Δ(**v-vector**)/Δt. a, a_c and a_T are just convenient categories that describe the $\overrightarrow{\Delta v}/\Delta t$ for three common situations.
The difference between these a's is explored later, in Part 3 of Chapter 5D.

Chapter 5B: Centripetal Acceleration

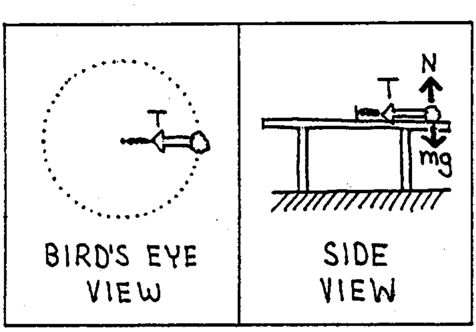
As just described, when an object moves along a circular path it has "centripetal" acceleration, a_c. Analysis of Δv/Δt (this is done in Section 5.93) shows that
a_c <u>magnitude</u> is $v_T{}^2/r$, where v_T is tangential speed and r is the circle's radius, and
a_c <u>direction</u> is always toward the circle's center, ⊥ to the v_T direction at that instant.

F (and thus a) that is directed along the line of motion will cause v-magnitude [speed] to change*, but it can't make an object move along a curving path. On the other hand, F (and thus a) that is ⊥ to motion will cause v-direction to change, but it won't change v-magnitude*. * These two principles ($F_{||}$ will change speed, but $F_\perp$ won't) are emphasized in Section 4.2.

PROBLEM 5-A: A Horizontal Circle.
As shown in this bird's-eye view, a 1.25 meter string makes a .40 kg rock move in a circle with constant 2.0 m/s speed, on top of a 150 cm high horizontal frictionless table.What is the string tension?
If the string breaks when the rock is at the spot marked "•", how far from the table's edge will it travel (distance & direction) before striking the ground?



SOLUTION 5-A
Use F=ma for the centripetal direction: "T" is the only force toward the center of the circle, and $a_c = v_T{}^2/r$. A *bird's eye view* shows the rock's circular motion, but a *side view* is needed to show the vertical forces or the motion after the string breaks.



for the centripetal direction

$F_c = m\ a_c$

$F_c = m\ \frac{v_T{}^2}{r}$

$+T = (.40)\frac{(2.0)^2}{1.25}$

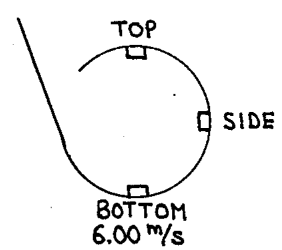
$T = 1.28\ n$

After the string breaks, the net force acting on the rock is zero; Newton's First Law states that the rock continues to move with its pre-release velocity of 2.0 m/s eastward until it reaches the table edge. If just-after-rock-leaves-table is "i" and just-before-impact is "f", we know 3-of-5 tvvax for the y-direction [horizontal means $v_i = 0$; $a = -9.8$; $\Delta x = -1.50$] and can find $\Delta t_y = .55$ s. Then use the *t-link* ($\Delta t_x = \Delta t_y$) for the x-direction equation: $\Delta x = v_x \Delta t_x = (2.0 \text{ m/s})(.55 \text{ s}) = 1.1$ m due east.

To continually change an object's v so it moves in a circle, *centripetal force* (F that points toward the circle's center) is required. For a rock on a string, F_c is obvious: the string's T-pull. But what causes the F_c when a car drives around a race track? {The answer is given later.} How many different circular motion situations can you think of? For each of these, determine what causes the "centripetal force".

PROBLEM 5-B: A Vertical Circle.

A .400 kg block slides down a frictionless ramp; the curved part is circular, with a diameter of 140 cm. At the bottom of the ramp, the block's speed is 6.00 m/s. Draw the block's F-diagram at the 3 positions shown.



If your class hasn't studied "Work-Energy" yet, skip the rest of this problem (come back to it later!) and read "Causes of Centripetal Force". What N-force does the ramp exert on the block at the bottom? at the side? at the top?

If the block moves around the ramp without losing contact, what is the slowest v_T it can have at the top?

SOLUTION 5-A

The force diagrams for bottom, side and top are shown below.

In Chapters 2, 3 and 4B, we split equations into x & y components. Just like F=ma can be split into $F_x = ma_x$ & $F_y = ma_y$, it can be split into $F_c = ma_c$ & $F_T = ma_T$.

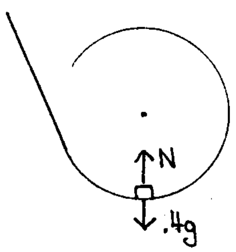
For substitution into $F_c = m a_c$, F_c is + if it points toward the center, and – if it points away from the center. If a force points in the tangential direction ($\perp$ to the radius-line), the F_c it contributes is zero. Study the F-diagrams and F=ma's. Do you agree with all F_c substitutions? Do you understand why each mg and N is +, – or 0?

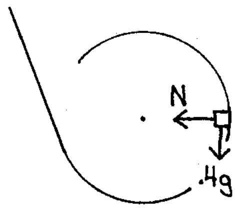
If necessary, a force can be "split" into centripetal and tangential components; this is done in Problem 5-#. Do you see why the F's at bottom, side & top don't need to be split in Problem 5-B?

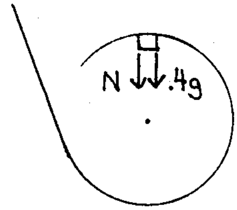
The direction of N changes constantly (it is ↑, then ← and ↓), but in another way N is constant because it always points toward the circle's center. In contrast, mg always points ↓, but its centripetal magnitude keeps changing (it is zero at the sides, maximum at top & bottom), and its centripetal direction flip-flops (it is – during the bottom half of the loop, but + during the top half).

When an object's height changes, it is a clue to use the Total Work Equation. The bottom-to-side TWE is $mg(0) + \frac{1}{2}m(6.0)^2 = \frac{1}{2}mv_{side}^2 + m(9.8)(+.70)$; the loop's diameter is 1.40 m, so its radius is .70 m. This TWE can be solved for $v^2 = 22.3$ at the side. The bottom-to-top TWE, with $\Delta h = +1.40$, gives $v^2 = 8.56$ at the top.

Do you see why two arithmetic steps are avoided if you solve for v^2 (which can be substituted directly into $F = mv^2/r$) instead of solving for v?







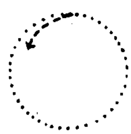
$$F_c = m\frac{v_T^2}{r}$$
$$+N - .4(9.8) = .4\frac{6.0^2}{.70}$$
$$N = +3.92 + 20.6$$
$N = +24.5$ *Newtons*

$$F_c = m\frac{v_T^2}{r}$$
$$+N + 0 = .4\frac{22.3}{.70}$$
$N = 12.7$ *N*

$$F_c = m\frac{v_T^2}{r}$$
$$+N + .4g = .4\frac{8.56}{.70}$$
$N = .97$ *N*

Why is N larger at the bottom than at the top? { In Problem 5-##, two reasons are discussed. }

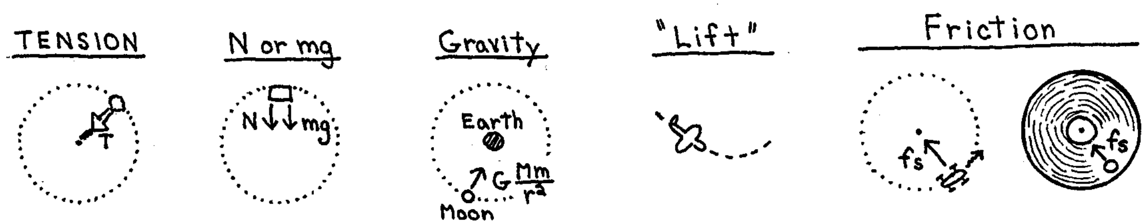
When $v_{top} = \sqrt{8.56} = 2.93$ m/s, $N_{top} = .97N$. If the loop is continuous (so it is O, not) and there is a small amount of friction, v_{top} decreases with each revolution. As v_{top} continues to ↓, N_{top} also ↓ until it becomes zero. { N drops to zero at the top before it is zero at any other point in the circle. Instead of trying to "prove" this, I'll just ask you to accept that it's true.} At the critical N=0 point, $F_c = ma_c$ is "$0 + mg = mv_T^2/r$", which gives $v_T = 2.62$ m/s. If v_T at the top is less than 2.62 m/s, $mg = mv_T^2/r$ gives an "r" that is less than .70 m. This means that the block won't travel around the full circle, but travels along a non-circular path like the - - - - shown below:



CAUSES OF CENTRIPETAL FORCE

The pictures below show some forces that can cause circular motion. T, N and mg are illustrated in Problems 5-A & 5-B. The force of gravity (which can cause a_c and circular motion) extends to outer space; this is studied in Chapter 5C, "Gravity, GMm/r^2, Orbiting Planets". An airplane can change direction or fly in circles because of the "lift force" generated by its wings.

If a road-curve is covered with frictionless ice, a car will slide off the road, like Problem 5-A's rock when its string breaks. On a normal road, friction between the tires and road provides the F_c (and thus a_c) that lets the car change its v-direction as it moves around the curve. Similarly, friction lets a penny on top of a rotating phonograph record ride in a circle without sliding. { Problems 5-# to 5-# involve friction on horizontal and "banked" roads, a phono record and a rotating funnel. }



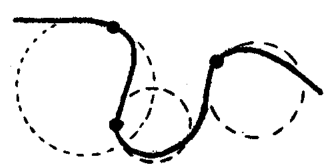
CAUSE-EFFECT: $F_c = m\,a_c$ is a "cause → effect equation". A centripetal force "F_c" causes "m" to move along a circular path with an acceleration of "a_c".

PSEUDO-FORCES: *Centripetal is a direction* (like x or y), not the name for a new kind of force. Real objects cause real forces (like gravity, rope tension, N-push, or friction) that can be drawn on a F-diagram. But there can never be a force caused by "centripetal", just like "x" or "y" don't cause force.

Similarly, velocity and acceleration don't produce force. { When your car turns a corner, you seem to feel your body being pulled toward the outside of the curve by what is often called "centrifugal" (not centripetal) force. Is this a real force, or illusion? This is discussed in Problem 5-#: Centrifugal Force.}

RADIUS OF CURVATURE

If an object travels around only part of a circle, you can still find the magnitude and direction of a_c at a certain place. Just draw a "whole circle" that matches the shape of the object's curving path at that point; this is done for the 3 •-points below. Then use the center & radius of this *radius-of-curvature* circle to find the a_c vector's direction (toward the center of the circle) and magnitude (v^2/r).



5C: Gravity, GMm/r², Orbiting Planets

Part 1: Gravity Force = GMm/r^2

In careful experiments, it has been observed that all objects attract each other with *gravitational force.* $F_{gravity}$ has a MAGNITUDE of GMm/r^2, where G is a constant-of-nature with a value (in SI units) of 6.67×10^{-11}, M & m are the objects' masses, and r is the distance between the objects' centers. The DIRECTION of $F_{gravity}$ is an attractive pull, directed on a line between their centers.

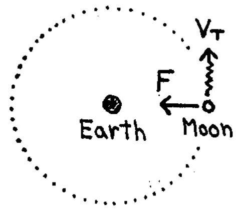
Near the earth's surface, the magnitude of $F_{gravity}$ can be described by either "$F_{gravity} = mg$" (where g is approximately 9.80 m/s²) or "$GM_e m/r_e^2$" (where M_e & r_e are the mass & radius of the earth). Compare these formulas. Does it look like g equals GM_e/r_e^2? This is almost true, but not exactly; the small differences between g and GM_e/r_e^2 are explored in Problem 5-#.

Chapter 5C covers only the "essentials" of gravity & orbiting. Your teacher and text can fill in some of the interesting details about astronomy & scientific history. Also, Problems 5-# through 5-## cover a variety of topics: Kepler's Second & Third Laws, the earth's rotation, ratio logic, $PE_{gravity} = -GMm/r$ (optional), and more.

OPTIONAL: Section 18.32 shows how to use calculus to solve $F_{gravity}$ problems.

Part 2: GMm/r^2 can cause Centripetal Motion: the Orbiting of Planets and Satellites.

As emphasized in Chapter 5B, circular motion is caused by *centripetal force* that is directed toward the center of a circle. The picture below shows how gravity can provide this force; notice that the GMm/r^2 gravity-force points toward the earth, which is at the center of the orbit-circle. Examples of *orbiting* are a planet (like our earth) circling* around the sun, and the moon or an artificial satellite moving around the earth. *Orbiting objects actually move in paths that are ellipses, not circles, but the same rotational-motion principles still apply. Elliptical orbits are discussed in "Kepler's Laws", Problem 5-#.}



To get the first formula below, substitute $F_c = GMm/r^2$ and $a_c = v_T{}^2/r$ into $F_c = ma_c$. The other formulas are derived in Problem 5-#, using simple tools from Chapter 5D.

$$\frac{GM\cancel{m}}{r^2} = \cancel{m}\frac{v_T{}^2}{r} \qquad \frac{GM}{4\pi^2}T^2 = r^3 \qquad GM = r^3\,\omega^2$$

where $G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2/kg^2}$ (in SI units), **M** is the mass of the object in the center, **m** (which cancels) is the mass of the orbiting object, **r** is the center-to-center distance between the objects, v_T and ω are the orbiter's tangential and angular speeds (in m/s and radians/s), and **T** is the time it takes the orbiter to make 1 revolution. (If you haven't read Chapter 5D yet, you can ignore the third formula and "ω".)

PROBLEM 5-C: How to Use the Orbiting-Object Formulas

Part 1: If a 150 kg satellite stays directly above the same spot on the earth during its orbit, how high is it above the earth's surface? How fast is it moving? {The earth's mass and radius are 5.98×10^{24} kg and 6.37×10^3 km.}

Part 2: The moon circles the earth once every 27.32 days. If a satellite which is 4.225×10^7 m from the earth's center orbits in 1.00 day, how far is it from the earth's center to the moon? { Pretend that you must use "ratio logic" because you don't know m_{earth}. }

SOLUTION 5-C

Part 1: The satellite must "match" the earth-spot's rotation and make 1 revolution every 24 hours, so $T_{satellite}$ = (24 hours)(3600 s/hr). As shown below, you can solve for "$r = 4.225 \times 10^7$", which is the distance from the satellite-center to earth-center.

$$\frac{GM}{4\pi^2}\,T_{sat}{}^2 = r_{sat}{}^3$$

$$\frac{(6.67\times 10^{-11})(5.98\times 10^{24})}{4\pi^2}(24\times 3600)^2 = r_s{}^3$$

$$4.225\times 10^7 \text{ meters} = r_s$$

To answer "... how high is it above the earth's surface?", subtract the earth's radius to get a height of $4.225\times10^7 - 6.37\times10^6 = 3.59\times10^7$ m = 35900 km = 22300 miles.

For 1 revolution, v_T = distance traveled / $\Delta t = 2\pi(4.225 \times 10^7)/86400 = 3070$ m/s, which is 6870 miles/hour. Or you can use equations from Chapter 5D to get the same answer: $v_T = r\omega = r[2\pi f] = r[2\pi(1/T)]$.

Part 2: Several "ratio logic" strategies are discussed in Section 19.9. One method is shown below; when the moon-equation is divided by the satellite-equation (why is this an acceptable algebra operation?), M_e cancels and you can solve for r_{moon}.

$$\frac{\cancel{\left(\frac{4\pi^2}{GM}\right)} T_m^2}{\cancel{\left(\frac{4\pi^2}{GM}\right)} T_s^2} = \frac{r_m^3}{r_s^3}$$

$$\frac{(27.32)^2}{(1.00)^2} = \frac{r_m^3}{(4.225 \times 10^7)^3}$$

$$3.83 \times 10^8 \text{ m} = r_m$$

When equations are divided like this, you can use non-SI units like "days", but you must be consistent. You cannot, for example, use 27.32 days for t_{moon} and 24 hours for $T_{satellite}$.

A memory-trick: to remember the T & r exponents for ratio logic (as used above), think-and-hear "T-two , r-three".

5D: Rotational Motion tvvax Analogies

Part 1: Rotational Distance, Velocity and Acceleration.

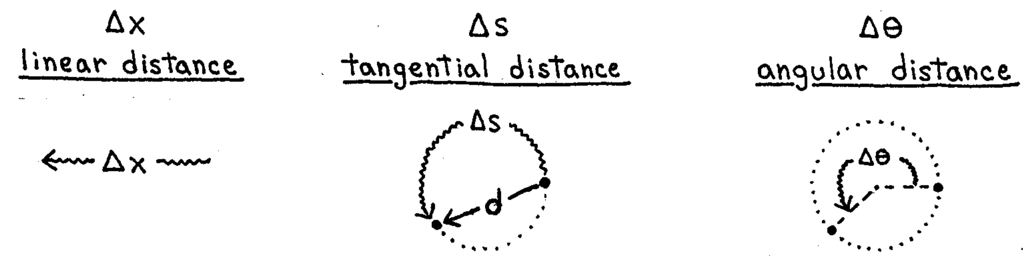
To help you understand the rotational analogies to "tvvax" linear motion, we'll continue the race car example from Section 5A.

If a car drives 30 m/s for 25 seconds in a straight line, it travels a *linear distance* of $\Delta x = (30 \text{ m/s})(25 \text{ s}) = 750$ m. If it drives around a circular track at 30 m/s for 25 s, it still travels 750 m; this around-the-circle distance is *tangential distance*, Δs. In the middle picture below, notice the difference between Δs (the actual distance the car travels*) and d (the *displacement vector* made by drawing a straight line from i to f).

* The change in the car's "odometer" (its mileage-meter) is Δs.

In each example above, the car's speedometer reads 30 m/s. If motion is in a straight line, this is *linear speed* ($v = 30$ m/s). If motion is along a circular path, it is *tangential speed* ($v_T = 30$ m/s).

The car's movement can also be described in terms of *angular distance*. If the distance around the circle is 1200 m, 750 m is 5/8 revolution or (because 1 rev = 360 degrees) 225 degrees; this angular distance is abbreviated "$\Delta\theta$".



3 kinds of velocity are defined by dividing distance (either linear "Δx", tangential "Δs", or angular "$\Delta\theta$") by Δt. These are abbreviated v, v_T and ω, respectively.

Similarly, 3 kinds of acceleration are defined by dividing Δvelocity (either linear "v", tangential "v_T", or angular "ω") by Δt. These are abbreviated a, a_T and α.

Here are the six **DEFINING EQUATIONS** for velocity & acceleration:

	LINEAR	TANGENTIAL	ANGULAR
distance	Δx	Δs	$\Delta\theta$
velocity	$\frac{\Delta x}{\Delta t} \equiv v$	$\frac{\Delta s}{\Delta t} \equiv v_T$	$\frac{\Delta\theta}{\Delta t} \equiv \omega$
acceleration	$\frac{\Delta \omega}{\Delta t} \equiv a$	$\frac{\Delta v_T}{\Delta t} \equiv a_T$	$\frac{\Delta\omega}{\Delta t} \equiv \alpha$

Later, Problem 5-D illustrates most of the ideas and equations in Chapter 5D.

Part 2: Radians and Connecting-Equations

RADIANS are needed to conveniently describe rotational motion in equations.

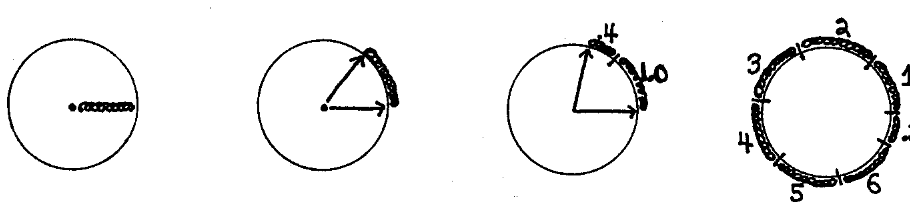
This 4-step "visual explanation" will help you understand what a radian is:

1) Cut a string (⟿ in the picture below) to the exact length of the circle's radius.

2) Imagine that the circle has a low rim, so you can stretch the radius-string around it as shown in the second picture. Now draw lines from the circle's center to the end-points of the string. The angle between these two lines is defined to be a *radian,* abbreviated "rad".

3) To get an angle of 1.4 radians, stretch 1 whole string and .4 of another string around the rim. Then draw lines from the circle-center to the string's end-points, as shown in the third picture. The angle between these lines is 1.4 radians.

4) The distance around a circle (its *circumference*) is $2\pi r$; $\pi = 3.14159...$, so $2\pi r$ is approximately $6.28\,r$. 2π radius-strings will go once around the circle, so <u>1 revolution = 2π rads = 360 degrees</u>. Dividing 360 by 2π gives <u>1 rad = 57.30 degrees</u>.



If the third circle above has a radius of 5.0 m, the angle of 1.4 radians (made by stretching 1.4 of the 5-meter radian strings along the rim) covers a distance of 7.0 m.

In the following equation, $\Delta\theta$ must be in radians if the units are to cancel properly.

$$1.4 \text{ radians} \quad \frac{5.0 \text{ meter}}{\text{radian}} \quad = \quad 7.0 \text{ meters}$$

$$\Delta\theta \quad r \quad = \quad \Delta s$$

$\Delta s = r\,\Delta\theta$ shows the "connecting relationship" between tangential and angular distance, so I call it a *connecting equation.* To derive the analogous connecting-equations for velocity and acceleration, just divide both sides of $\Delta s = r\,\Delta\theta$ by Δt and use the definitions of v_T & ω, then divide again by Δt and use the a_T & α definitions. [These derivations are done in Section 5.93.]

Important: Use only radians for $\Delta\theta$, ω and α in these <u>CONNECTING EQUATIONS</u>:

$$\Delta s = r\,\Delta\theta$$
$$v_T = r\,\omega$$
$$a_T = r\,\alpha$$

Δs, $\Delta\theta$, v_T, ω, a_T and α each appear in a defining-equation (Part 1) and in a connecting-equation. You can use these any of these 6 variables for *solve-and-use links.*

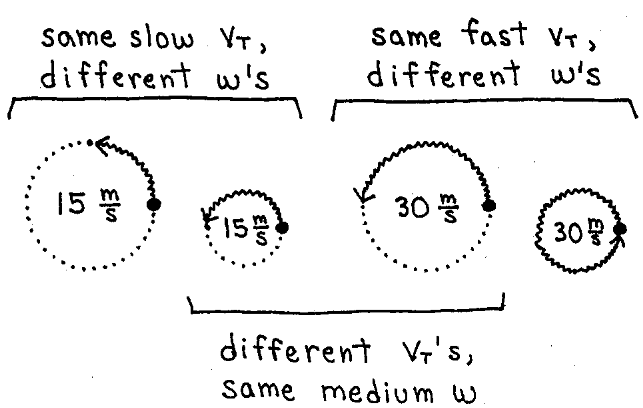
Part 3: Comparing 2 kinds of Velocity, and 4 kinds of Acceleration.

The difference between **TANGENTIAL** and **ANGULAR** VELOCITY:

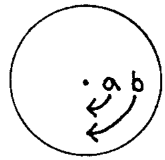
Four cars have a 1-lap race, as shown in the 4 pictures below. Two cars are slow [15 m/s] and two are fast [30 m/s]. They race on a large track [r = 382 m, circumference = $2\pi r$ = 2400 m] and a small track [r = 191 m, circumference = 1200 m]. The pictures below show the position of each car at the finish, the instant the fourth car (which wins because it is going fast on a small track) finishes its lap after 40 seconds. Compare the v_T's and ω's of each car. Do you see why identical v_T's doesn't necessarily mean identical ω's? (The car with the largest ω wins this 1-lap race.)

CALCULATIONS. The third car's ω can be calculated with a connecting equation: $\omega = v_T / r = (30 \text{ m/s})/(381 \text{ m/radian}) = .0785$ radians/s. Or you can solve the ω-defining equation: $\omega \equiv \Delta\theta/\Delta t = (.5 \text{ rev})(2\pi \text{ rads/rev})/(40 \text{ s}) = .0785$ radians/s.

If you want, calculate the second car's ω, and you'll see why it is also .0785 rads/s. Using "ratio logic", do you see why the second and third cars have the same ω?



Points a & b on this spinning plate have the same ω (they make 1 revolution in the same time), but different v_T's (b travels further during the revolution, so it is faster):

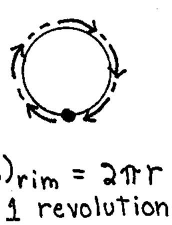


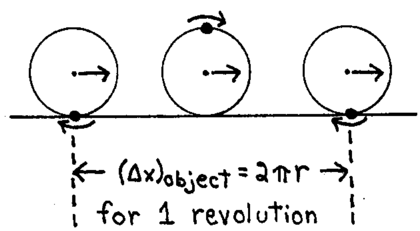
This relationship can be explained using connecting-equation "ratio logic":
$v_T = r\,\omega$, and ω is the same for both points, so v_T increases as r increases.

COMBINED MOTION: LINEAR + ROTATIONAL

Watch the tires of a forward-moving car; the tires **rotate** as they **move forward.** Compare the forward-only movement of the entire car (which can move around a circle-track even though it doesn't "rotate" by tumbling end over end) and the tire's "forward-and-rotating" motion. Do you see the important difference between the car-motion and the tire-motion?

In the first picture below, during 1 revolution of the tire a "rim point" • moves once around the tire-circumference: • moves through an angle of 2π radians, and has $\Delta s = \underline{2\pi}\, r_{rim} = \underline{\Delta\theta}\, r_{rim}$. The right-side pictures show a tire at the start, middle & finish of one revolution. Watch the contact-surface of a rolling object, think about what is happening, and convince yourself that the tire will move forward a distance equal to the Δs of a rim-point: during one revolution, $(\Delta x)_{tire} = (\Delta s)_{rim} = r_{rim}\,\Delta\theta = 2\pi\, r$.





Notice that Δs_{rim} is the "rotational distance" moved by the rim-point •. When the tire is rolling, Δs_{rim} doesn't include the sideways movement of • as it is carried along with the tire toward the right.

{Another interpretation of combined movement is given in Problem 5-1#, which shows an easy way to calculate the "instantaneous total velocity" (rotation-v + sideways-v) of a rim-point.}

Dividing "$(\Delta x)_{object} = (\Delta s)_{rim} = r_{rim}\,\Delta\theta$" by Δt and using definitions (for v, v_T, ω) gives the analogous relationships for velocity that are shown below. Dividing by Δt again and using definitions (for a, a_T, α) gives the "acceleration relationships".

For <u>**NON-SLIP ROLLING**</u>, with all angles measured in radians,

$$\begin{aligned} (\Delta x)_{object} &= (\Delta s)_{rim} = r_{rim}\,\Delta\theta \\ v_{object} &= (v_T)_{rim} = r_{rim}\,\omega \\ a_{object} &= (a_T)_{rim} = r_{rim}\,\alpha \end{aligned}$$

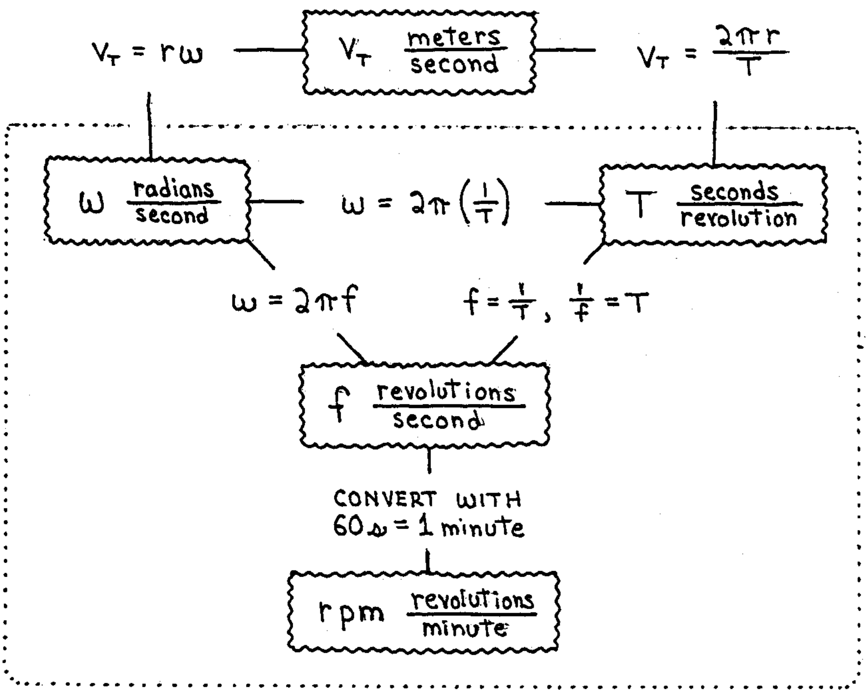
These equations are valuable; they are used for problem-solving in Chapter 5F.

ANGULAR VELOCITY UNITS

Four ways to measure angular velocity are ω (in radians/s), **f** (in revolutions/s)*, **rpm** (revolutions per minute), and {a unit that isn't used often} in degrees/s. The time-per-revolution [**T**, in seconds/revolution] isn't an angular speed, but it expresses the same information. *another symbol for frequency is "υ"

These five can be interconverted by using the conversion factors "1 revolution = 2π radians = 360 degrees" and "60 seconds = 1 minute", plus the fact that f (in revs/second) can be flipped upside down to get T (in seconds/rev).

The equations below can be derived by playing with links, definitions & conversion factors. Notice that each equation inside the {....} has only 2 variables; if you know either of them, you can find the other. <u>If you know any 1 of these 4 (ω, f, rpm, T), you can find all of the others</u>, so ω/f/rpm/T is really just 1 variable, not 4. And if you know 2 of these 3 (v_T, r, ω/f/rpm/T), you can find the third.



FOUR KINDS OF ACCELERATION

are linear (a), tangential (a_T), angular (α) and centripetal (a_c).

The summary below will help you understand their similarities and differences.

The first part of the summary is SITUATIONS. Three important types of motion are discussed at the end of Chapter 5A: linear (with only "a"), constant-speed circular (only a_c), and changing-speed circular (a_T and α, a_c).

F=ma EQUATIONS: "$F_c = m\, a_c$" is used to solve problems in Chapter 5B. Later, in Chapter 5F, $F_T = m\, a_T$ and $\tau = I\, \alpha$ are discussed.

MAGNITUDE: Can you find 3 *defining equations* and 1 *connecting equation* in the summary? Here is an easy derivation: $a_c = v_T{}^2/r = (r\omega)^2/r = r^2\,\omega^2/r = r\,\omega^2$.

{ For 2-D situations, $\Delta v/\Delta t \equiv a$ can be split into x & y components: $\Delta v_x/\Delta t = a_x$ and $\Delta v_y/\Delta t = a_y$. }

DIRECTIONS are discussed in Chapters 5A and the start of 5B. A principle from Section 2.2 is true for v-and-a, v_T-and-a_T, ω-and-α; if speed is increasing **v** and **a** point in the same direction, but if speed is decreasing **v** and **a** point in opposite directions.

Optional: 1) To get the "total acceleration vector" for circular motion, add the a_C & a_T components (which always point $\perp$ to each other) as vectors. 2) For most purposes, you can consider α to be in the same direction (with the same $\pm$ sign) as a_T, but the "real direction" of the α-vector is discussed in Chapter 5E's Step 4.

WHAT IS CHANGING? For a & a_T & α, the magnitude (of v & v_T & ω) is changing. But for a_C it is v-direction that is changing, not v-magnitude.

		a LINEAR	a_T TANGENTIAL	α ANGULAR	a_c (or a_R) CENTRIPETAL
SITUATIONS	LINEAR MOTION	a is "REGULAR" ACCELERATION (as in Chapters 2-4)	not used	not used	0
	CONSTANT-SPEED CIRCULAR MOTION	not used	0	0	a_c shows the rate-of-change of $\vec{v}$-direction as object moves along a curve.
	CHANGING-SPEED CIRCULAR MOTION	not used	a_T shows v_T's rate-of-change	α shows ω's rate-of-change	
	F = ma EQUATION	$F_x = m a_x$ $F_y = m a_y$	$F_T = m a_T$	$\tau = I \alpha$	$F_c = m a_c$
	MAGNITUDE	$\frac{\Delta v}{\Delta t}$	$\frac{\Delta v_T}{\Delta t}$ $a_T = r \alpha$	$\frac{\Delta \omega}{\Delta t}$	v_T^2 / r and $r \omega^2$
	DIRECTION	If speed ↑, out "front window". If speed ↓, out "rear window". {Along direction of motion.}		α has same $\pm$ sign as a_T. {see * below}	Toward center of circle (on radial-line)
	What is changing?	Magnitude of v	Magn. of v_T	Magn. of ω	Direction of v vector

Part 4: A "tvvax System" for Rotational Motion

The tvvax system, a simple yet powerful tool for analyzing linear motion, can also be used for rotational motion. Just do the following 3-step transformation: turn a tvvax equation (with Δt, v, a, Δx) into a tvvas equation (with Δt, v_T, a_T, Δs), make all possible connecting-equation substitutions ($\Delta s = \Delta\theta\, r$, $v_T = \omega r$, $a_T = \alpha r$), cancel the r's.

If you do this for each tvvax equation, you'll get 5 analogous "$t\omega\omega\alpha\theta$" equations:

$v_f - v_i = a t$	$\Delta x = \left(\frac{v_i + v_f}{2}\right) t$	$\Delta x = v_i t + \frac{1}{2} a t^2$	$\Delta x = v_f t - \frac{1}{2} a t^2$	$v_f^2 - v_i^2 = 2 a (\Delta x)$
$\omega_f - \omega_i = \alpha t$	$\Delta\theta = \left(\frac{\omega_i + \omega_f}{2}\right) t$	$\Delta\theta = \omega_i t + \frac{1}{2} \alpha t^2$	$\Delta\theta = \omega_f t - \frac{1}{2} \alpha t^2$	$\omega_f^2 - \omega_i^2 = 2 \alpha (\Delta\theta)$

The overall result of these derivations is that every linear-motion variable has been replaced by (even though it is not "equal to") the analogous angular-motion variable. This transforms "pure linear" tvvax equations into "pure angular" $t\omega\omega\alpha\theta$ equations. The difference between *replacement* and *substitution* is important:

Δs is replaced by $\Delta\theta$, but Δs equals $r\,\Delta\theta$.
v_T is replaced by ω, but v_T equals $r\,\omega$.
a_T is replaced by α, but a_T equals $r\,\alpha$.

tvvax strategies (as in Section 2.4 or the Chapter 2 Summary) can be used for $t\omega\omega\alpha\theta$: Read/think/draw, choose i & f points for a constant-α interval, make a $t\omega\omega\alpha\theta$ table, look for 3-of-5, choose a 1-out equation, substitute and solve, answer the question.

UNITS: "ω" is used in writing the $t\omega\omega\alpha\theta$ equations; this seems to imply rads/s, but other units can be used for angular velocity. **Most rotational motion equations** (connecting-equations, Chapter 5F analogies to F=ma & Work-Energy & Impulse-Momentum, ...) **require the use of radians.** But for $t\omega\omega\alpha\theta$, just be consistent:

You can use all radians & seconds [s , rads/s , rads/s^2 , rads]*, or
you can use all revolutions & seconds [s , revs/s , revs/s^2 , revs]*, or
you can use all revolutions & minutes [mins, revs/min, revs/min^2, revs], or
you can use all degrees & seconds [s , degs/s , deg/s^2 , degs],
but you can't mix rads with revs (or degrees), or seconds with minutes.
{ * These are the two most common unit-combinations for $t\omega\omega\alpha\theta$. }

PROBLEM 5-D: Using the Rotational Motion Equations

Starting from rest, a plate with 160.0 cm diameter takes 10.0 seconds of constant acceleration to reach 240 rpm. During this 10 s, what is α (in rads/s), and how many revolutions does the plate make?

At the end of this 10 s, what is ω, T (time per rev), v_T and a_c for a point that is 10 cm in from the outer edge? Are these answers different for a point on the outer rim?

SOLUTION 5-D:

From 0 to 10 s

$\Delta t = 10\text{ s}$
$\omega_i = 0$
$\omega_f = 240 \frac{\text{revs}}{\text{min}} \left(\frac{1\text{ min}}{60\text{ s}}\right)$
$\alpha =$
$\Delta\theta =$

Using "$\Delta\theta$ out"

$\omega_f - \omega_i = \alpha\,\Delta t$
$\frac{240}{60} - 0 = \alpha\,(10)$
$.40\ \text{revs/s}^2 = \alpha$
$\left(\frac{2\pi\text{ rads}}{1\text{ rev}}\right).40\ \text{revs/s}^2 = \alpha$
$2.51\ \text{rads/s}^2 = \alpha$

Using "α out"

$\Delta\theta = \frac{1}{2}\,(\omega_i + \omega_f)\,\Delta t$
$\Delta\theta = .5\,(0 + 4.0\,\frac{\text{revs}}{\text{s}})(10\text{ s})$
$\Delta\theta = 20$ revolutions

These 2 questions involved an <u>interval</u>. The next 4 questions ask "What is happening at one <u>instant</u>?".

The plate has $r = \frac{1}{2}(1.60\text{m}) = .80$ m, so a point 10 cm in from the rim has $r = .70$ m.

$$\omega = \frac{240\text{ revs}}{\text{min}}\ \frac{1\text{ min}}{60\text{s}}\ \frac{2\pi\text{ rads}}{1\text{ rev}} = 25.1\,\frac{\text{rads}}{\text{s}}$$

$$T = \frac{1}{f} = \frac{1}{(240/60)\text{ revs/s}} = .25\text{ s}$$

$$\begin{aligned} V_T &= r \quad \omega \\ &= r \quad 2\pi f \\ &= (.70)(2\pi)(4.0) \\ &= 17.6 \ m/s \end{aligned}$$

$$\begin{aligned} a_c &= \frac{V_T^2}{r} \\ &= \frac{(17.6)^2}{.70} \\ &= 443 \ m/s^2 \end{aligned}$$

$$\begin{aligned} a_c &= r \quad \omega^2 \\ &= (.70)(25.1)^2 \\ &= 441 \ m/s^2 \end{aligned}$$

{ROUNDOFFS CAUSE $441 \neq 443$}

For a point on the rim, the "instantaneous ω" is still 25.1 rads/s, $v_T = r\omega = (.80)(25.1) = 20.1$ m/s, and a_c can be calculated by either $a_c = r\omega^2 = .80(25.1)^2 = 505$ rads/s, or by $a_c = v_T{}^2/r = (20.1)^2/.8$.

5E: Torque

This sub-chapter shows how to calculate *torque,* abbreviated τ.
Chapters 5F & 5G show how to use torque for solving problems.

Torque is the ability of an applied force to change an object's rotational velocity; the relationship of force and torque is discussed in detail at the start of Chapter 5F.

To calculate the torque acting on an object, use the following 5-step process.

Step 1: <u>Draw a F-diagram</u>, showing all forces that push or pull on the object. You must draw each F at the place where it is actually applied to the object.

Step 2: <u>Choose a τ-axis</u>. { τ is always calculated "with respect to" a specific axis.}

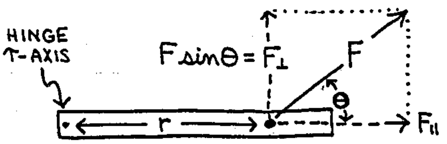
Step 3: <u>Use a τ-formula</u>. In the bird's eye pictures below, a door rotates on its hinge (this hinge is the obvious choice for τ-axis) and is pulled by a force "F". Two τ-formulas can be used. Here are explanations of how to use these formulas:

For $\tau = \mathbf{r\,F\,\sin\theta}$, $\vec{r}$ is a line drawn from the τ-axis to the point where F is being applied, and θ is any angle between F and r (or their extensions). Look at the first picture below, and imagine you are pulling the door with the $F_{||}$-component. Can you make the door rotate? No. But the door will rotate if you pull it with $F_\perp$, because **torque is only produced by F that is $\perp$ to the r-vector**. As shown in the derivation below, "$\tau = \mathbf{r} \ \ \mathbf{F_\perp}$" is the foundation equation for "$\tau = r\,F\,\sin\theta$".

Or use $\tau = \mathbf{r_\perp} \ \mathbf{F}$. To find $r_\perp$ (which is often called the *lever arm* or *moment arm),*

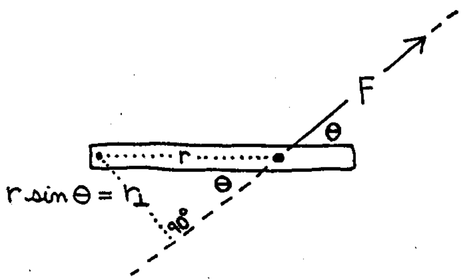
a) Draw the extensions of F, as shown by the ---- lines on the right picture below.
b) Ask "Where does this F-extension come closest to the τ-axis?", and answer "When the axis-to-extension line (shown by ····) makes a 90° angle with the extension.
c) The length of ····, the shortest axis-to-extension line, is $r_\perp$.

$\tau = F\,r\,\sin\theta$



$$\begin{aligned} \tau &= r \quad F_\perp \\ \tau &= r \ (F\sin\theta) \\ \tau &= r \ F \sin\theta \end{aligned}$$

$\tau = r_\perp\,F$



Do both methods give the same answer? Yes: $r_\perp$ is $r \sin\theta$, so $\tau = r_\perp F = r(F \sin\theta)$.

Step 4: <u>Determine the ± sign of torque</u> by using this 4-step process:

A) POINT your pen in the direction of r (from the τ-axis to the F-point).
B) HOLD one end of the pen down at the τ-axis.
C) PUSH OR PULL the pen (at the F-point) in the direction F points.
D) WATCH which way the pen rotates. If you have defined ↶ to be + (this is usually done, unless there is a good reason to define ↷ as +) and the pen rotates ↶ , τ is + ; if the pen rotates ↷ , τ is –.

OPTIONAL: If your class uses "vector cross products", or mentions a "right hand rule" or torque direction, read this paragraph and also Section 18.62. * Here is an easy *right-hand rule:* hold your right hand like this [ ], curl your fingers in the rotation-direction, and your thumb will point in the τ-direction. For example, ↷ rotation makes your right thumb (and τ) point downward toward the page. This same process is used to define the direction of an α-vector (α is studied in Chapter 5D) or L-vector (as studied in 5F). {For most physics calculations these "vector-directions" are not used, but they are needed to determine the *precession* direction of a spinning top or bicycle wheel; this is explained in Section 5.94.}

Step 5: If more than one torque acts on an object, τ_{total} = the sum of individual τ 's.

UNITS: Because $\tau = F r \sin\theta$, τ has units of "N m". {In Section 4.1, *work* is defined as $W \equiv Fd$. Work also has units of "Nm", but the "Nm" of W (which is a "Joule") is not the same as the "Nm" of τ. I won't try to explain the reasons for this difference, but will just ask you to accept it.}

A comment on Step 2: A τ-axis is actually a line, but in a 2-dimensional diagram it is drawn as a "point". For example, a door's hinge (the axis about which it rotates) is a vertical line; but in the bird's eye 2-D diagram above, this axis-line appears to be a point.

CHOICES: If your class studies how torque changes angular motion, move ahead to Chapter 5F. If you are using torque to study static situations, skip to Chapter 5G.

CHOICES: If you haven't read sub-Chapters 5A, 5D and 5E yet, do it now.
Study 5F's Part 1 first, then use the "CHOICES" suggestions at the end of it.

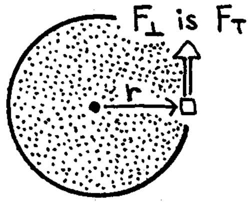
5 F: Torque Dynamics – Rotational Analogies for F=ma, Work-Energy, Impulse-Momentum.

Part 1: Rotational Inertia

Rotational inertia (or *moment of inertia),* abbreviated "I", is defined as $I \equiv m r^2$. Why? To find out, let's look at how F=ma can be used to analyze circular motion.

As discussed in Chapter 5B, *centripetal force* (F_c) makes an object move along a circular path. But F_c is ⊥ to the direction-of-motion, so F_c doesn't change an object's speed. Only *tangential force* (F_T) along the direction-of-motion will produce a_T that changes an object's v_T; this is expressed in the formula "$F_T = m\, a_T$".

The bird's eye picture below, of a disk rotating about its center, shows that F_T is ⊥ to the r-vector. The F_T in "$F_T = m\, a_T$" is the same as the $F_\perp$ in "$\tau = F_\perp r$", so the τ-formula can be written "$\tau = F_T\, r$"; this is used in transforming $F = ma$ into $\tau = I\alpha$.



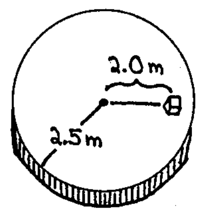
$$F_T = m \; a_T$$
$$\Downarrow \quad \Downarrow \quad \Downarrow$$
$$\left(\frac{\tau}{r}\right) = \left(\frac{I}{r^2}\right)(\alpha r)$$
$$\tau = I \; \alpha$$

In the transformation above, the r's neatly cancel, to turn F=ma into its angular-motion analogy $\tau = I\alpha$. Why do the r's cancel? Because I is a "fudge factor variable", defined as it is ($I \equiv mr^2$) for the purpose of making the r's cancel. Very clever.

Here are some analogies between linear and angular motion.
F causes linear acceleration "a", while τ causes angular acceleration "α".
m is an object's resistance to having its linear velocity changed: as m ↑ , a ↓ .
I is an object's resistance to having its angular velocity changed: as I ↑ , α ↓ .

To calculate I, use the following 4 principles.

1) TOTAL = SUM OF PARTS. For a system of several objects, I_{total} = sum of individual I's. For example, the system at the right is a small heavy box (240 kg) resting .5 m from the rim of a solid wood cylinder (400 kg, 2.5 m radius). As you would expect, the system's total I is:

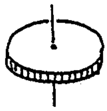
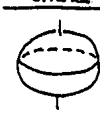
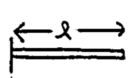
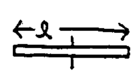
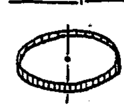


$$I_{total} = I_{box} + I_{cylinder}$$

2) FOR A "POINT OBJECT", $\underline{I = m r^2}$. A *point object* is so small that we can assume all of its mass is located at one point in space, at a distance "r" from the axis-of-rotation. {Like τ, I is always calculated "with respect to" a specific axis.}
Let's assume that the 240 kg box is small enough (relative to the size of r) to be considered a point object, so $I_{box} = mr^2 = (240 \text{ kg})(2.0 \text{ m})^2 = 960 \text{ kg m}^2$.

3) FOR A NON-POINT OBJECT, $\underline{I = X\,mr^2}$, where X is a "cut-down fraction" that depends on the object's shape. {All of the box's mass is 2.0 m from the axis. But the wooden cylinder has its mass spread out at different values of r. Some of its mass is close to the axis, some is far away, and most is in-between. The math technique of "calculus" splits the cylinder into millions of tiny masses, finds the mr^2 of each tiny mass, then adds these mr^2's together to get the cylinder's total I.} The results of such calculations are summarized in tables like the one below.

Notice that each formula fits the format of "$I_{total} = X\,mr^2$".

SOLID* CYLINDER	SOLID* SPHERE	THIN SPHERICAL SHELL *	THIN ROD ROTATING ABOUT END	THIN ROD ROTATING ABOUT CENTER	CYLINDRICAL SHELL * ("hoop")
					
$I = \frac{1}{2}mr^2$	$I = \frac{2}{5}mr^2$	$I = \frac{2}{3}mr^2$	$I = \frac{1}{3}m\ell^2$	$I = \frac{1}{12}m\ell^2$	$I = 1\,mr^2$

* SOLIDS and SHELLS: For a solid sphere, X = 2/5. But if a sphere is a hollow "shell", like a basketball with all mass at the outer edge, X = 2/3. {A cylinder can also be solid or a shell.}

These 3 principles (total = sum of parts; $I = mr^2$ or $I = Xmr^2$) can be usedto find the total rotational inertia of the box-and-cylinder system:

$$\begin{aligned} I_{total} &= I_{box} + I_{cylinder} \\ &= (240)(2.0)^2 + (\tfrac{1}{2})(400)(2.5)^2 \\ &= 960 + 1250 \\ &= 2210 \text{ kg m}^2 \end{aligned}$$

A third category: OTHER KINDS OF OBJECTS. If you encounter an object that doesn't fit into either of these categories (point or "standard Xmr^2 shape"), improvise — use common sense, calculus, ...

OPTIONAL: Two other ways to calculate I, the *parallel-axis formula* and *radius of gyration,* are discussed in Problems 5-## and 5-##. The Aesop's Problem in Section 18.33 shows two essential strategies for setting up integrals.

CHOICES: Use Parts 2 (Equation Overload), 3 (Angular F=ma: $\tau=I\alpha$), 4 (Angular Kinetic Energy & Work) and 5 (Angular Momentum & Impulse) in any order.

Part 2: How to Cope with "Equation Overload"

Chapter 5 has introduced many new variables (a_c, F_c, Δs, v_T, a_T, F_T, $\Delta\theta$, ω, f, rpm, T, α, τ, I) and equations and concepts. How can you make sense out of it and not be overwhelmed by "information overload"? Here are some useful suggestions.

The key is to **understand the equations** you are using.

1) What are the "If... , then..." requirements that tell you when it **can** be used?
Here are three examples. a) If [and only if] α is constant between an interval's initial & final points, then you can use the $t\omega\omega\alpha\theta$ equations. b) Chapters 5A & 5D compare closely related types of motion, to help you understand the different kinds of distance & velocity & acceleration, so you can decide which equation(s) can be used for a particular problem-situation. c) Part 1 of 5F splits objects into 3 categories: point objects ($I = mr^2$), basic shapes ($I = Xmr^2$), and "other objects".

2) What are the clues or strategies that tell you when it **should** be used?
Know and use equation-choosing tools like the 1-out strategy of Section 2.4 and the principles in Section 4.12. You can use these tools over & over again; for example, the 1-out tvvax strategy is used for $t\omega\omega\alpha\theta$, and 4.12's ideas can help you choose between the angular analogies to tvvax, F=ma, $F\Delta x=\Delta KE$ and $F\Delta t=\Delta mv$.

3) Learn the logical relationships and "variable links" between equations.
The chapter summaries (especially for Chapter 5) organize equations in a logical, visually meaningful way; this makes it easy to use equations for problem solving.
The importance of "linking" is discussed in Sections 2.3, 3.4, 4.1 and 5D. For example, if you know the equations that contain "**a**" (tvvax, F=m**a**, $\mathbf{a}_T = r\alpha$) and you need the numerical value of **a** to solve one of the equations, you'll know where to look for it — in one of the other equations that contains **a**.

4) What is the meaning of each variable?
Is it a "constant of nature" with a fixed value you can find in a table, or a variable? What **word** (or words) can be used in a problem-statement to describe it? What **units** does it have? What does it **look** like in a picture?

5) Learn (and remember) the "important details" that help you master an equation. Learn from your own discoveries, from problems that are worked out in this book, in your main text, and by your teacher. Use active review (of Flash Card Reviews and Chapter Summaries,...) so you'll remember what you've learned.

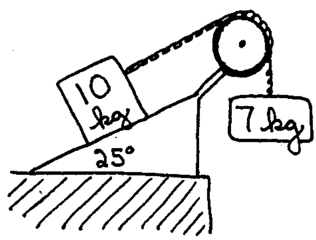
Part 3: Angular F=ma ($\tau = I\alpha$)

At the beginning of 5F's Part 1, some relationships between F=ma and τ=Iα were explored. Now we'll look at how F=ma & τ=Iα can be be used to solve problems.

In Chapter 3, pulleys were "massless". What can we do if a pulley does have mass?

PROBLEM 5-E: Using $\tau = I\alpha$ on a Real Pulley.

The pulley at the right is a solid cylinder, with an axle through its center and a small groove in its edge for the rope. It has a 30 cm radius and a weight of 100 *N*. There is enough friction between pulley and rope to keep the rope from slipping. The pulley bearings are frictionless, and so is the ramp.



Find the block-accelerations and rope-tensions. 1.5 s after starting from rest, how many revolutions has the pulley made? How far have the blocks moved?

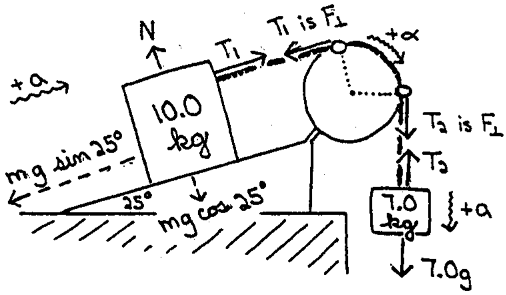
SOLUTION 5-E

MOTION: v_i = 0, and 7.0g is larger than 10.0g(sin 25°), so the 7 kg block moves downward and the pulley rotates ↷. The blocks, rope and pulley-rim have *matched motion;* as discussed in "Combined Motion, Linear + Rotational" (Part 3 of 5D), this means that $\underline{a_{blocks}} = a_{rope} =^* (a_T)_{rim} = \underline{r_{rim}\,\alpha}$. Notice the two underlined terms; the substitution of "$a_{blocks} / r_{rim} = \alpha$" is a key step in solving the equations below.

* This equal sign is true only if the rope pulls the rim ↷ with no slipping, thus causing the pulley-rim to move at the same speed as the rope.

TORQUE: In the F-diagram below, T_1 and T_2 are not equal. T_1 tries to turn the pulley ↶, opposite the rotation-direction I've chosen to be +, by producing a torque of $-.30\,T_1$ *. T_2 produces a ↷ torque of $+.30\,T_2$. The pulley accelerates ↷, so T_2 must be larger than T_1. * The rope pulls the pulley in the tangential direction (so T is $\perp$ to r) at the two "points of tangency" marked by o's. Because T and r are $\perp$, T is $F_\perp$ and r is $r_\perp$: all three τ-formulas [$\tau = rF\sin\theta$, $\tau = r\,F_\perp$, $\tau = r_\perp F$] give $\tau = Tr$.

ALGEBRA: 4 steps are shown below – 1, 2, 3, 4. In Step 1, both sides of the pulley equation are multiplied by 1/r (which is 1/.3), and all r's disappear! Step 2 adds all 3 equations together, as in Problem 3-B, cancelling the "internal" T's so you can solve for *a*. After Step 3's two substitutions, using "a = 1.23" in all possible places, one equation can be solved for T_2. Step 4: "use" T_2 and solve for T_1.



for the 10 kg block

$$F = ma$$
$$-10g\sin 25° + T_1 = 10a$$
$$\left\{\begin{array}{l} -T_1 + T_2 = 5.1a \\ -T_2 + 7g = 7a \end{array}\right\} ②$$
$$-10g\sin 25° + 7g = 22.1a$$
$$+1.23 = a$$

for the pulley

$$\tau = I\,\alpha$$
$$\left\{-.3T_1 + .3T_1 = \left[\tfrac{1}{2}\left(\tfrac{100}{9.8}\right).3^2\right]\tfrac{a}{.3}\right\} ①$$
$$-T_1 + T_2 = 5.1\,a$$
$$\Downarrow ④ \qquad \Downarrow ③$$
$$-T_1 + (60.0) = 5.1(1.23)$$
$$53.7 = T_1$$

for the 7 kg block

$$-T_2 + 7g = 7\,a$$
$$\Downarrow ③$$
$$-T_2 + 68.6 = 7(1.23)$$
$$60.0\ N = T_2$$

Knowing a = +1.23 gives 3-of-5 $t\omega\omega\alpha\theta$ [$\Delta t = 1.5$ s, $\omega_i = 0$, $\alpha = a/r = (1.23/.30)$ rads/s (remember that radians must always be used in connecting equations like $a = r\alpha$)]. To get $\Delta\theta$, solve the ω_f-out equation "$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$" for $\Delta\theta = 4.61$ rads, then answer the problem question by converting to revs: 4.61 rads (2π rads/rev) = .73 revs.

To find how far the blocks have moved, use $\Delta x_{blocks} = \Delta x_{rope} = \Delta s_{rim} = r_{rim}\,\Delta\theta$ = (.30 m/rad)(4.61 rads) = 1.4 m.

In the method above, we used $t\omega\omega\alpha\theta$ to get $\Delta\theta_{pulley}$ and then used $\Delta x = r\Delta\theta$ to find Δx_{block}. This order can be reversed. If you want, try it yourself: first solve tvvax for Δx, then use $\Delta x = r\Delta\theta$ to find $\Delta\theta_{pulley}$.

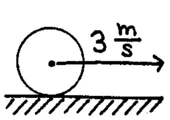
Part 4: Angular Work-Energy

This section assumes that you have read Chapter 4A (Work → Energy).

Let's continue the analysis of COMBINED MOTION (Linear + Rotational) that began in Chapter 5D's Part 3. In the first picture below, a spherical shell (mass 8.00 kg, radius .75 m) moves forward on a frictionless surface at 3.00 m/s, without rotating; this forward motion is called *translational motion.* In the next picture an identical ball spins on an axle at 4.00 radians/s, without moving forward; it has *rotational motion.* The last picture shows what happens when the ball rolls without slipping; as discussed in "Combined Motion", its translational-v (3.00 m/s) and rotational-ω (4.00 radians/s) are related by "$v_{object} = r_{rim}\,\omega$".

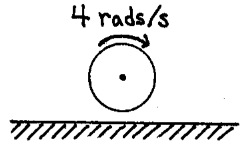
The first ball has kinetic energy caused by only translational motion: KE = $\frac{1}{2}mv^2$. The second ball has KE caused by only rotational motion: this is (using an equation that is derived soon) KE = $\frac{1}{2}I\omega^2$. The third ball has KE caused by both types of motion: $\underline{KE_{total}} = \underline{KE_{translation} + KE_{rotation}} = \underline{\tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2}$.

Translational KE



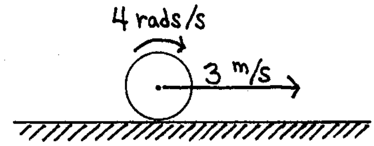
$$\tfrac{1}{2}\ m\ v^2$$
$$\tfrac{1}{2}(8)(3)^2$$
$$36\ J$$

Rotational KE



$$\tfrac{1}{2}\quad I \quad \omega^2$$
$$\tfrac{1}{2}\ \tfrac{2}{3}\,m\,r^2\ \omega^2$$
$$\tfrac{1}{2}\left[\tfrac{2}{3}(8)(.75)^2\right](4)^2$$
$$24\ \text{Joules}$$

(Translational + Rotational) KE



$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}\quad I \quad \omega^2$$
$$\tfrac{1}{2}(8)(3)^2 + \tfrac{1}{2}\left[\tfrac{2}{3}(8)(.75)^2\right](4)^2$$
$$36 \quad + \quad 24$$
$$60\ \text{Joules}$$

For an 8 kg rock moving around a circle of .75 m radius at 3 m/s (which is $\omega = v_T/r$ = 4 rads/s), KE can be calculated using either of these methods:

$$KE_{translation} = \tfrac{1}{2}\ m\ v^2 = \tfrac{1}{2}(8)(3)^2 = 36\ J$$

$$KE_{rotation} = \tfrac{1}{2}\quad I \quad \omega^2 = \tfrac{1}{2}[8(.75)^2](.4)^2 = 36\ J$$

This derivation shows why $\frac{1}{2}m(v_T)^2 = \frac{1}{2}I\omega^2$ for a *point object* like a "rock on a string":

$$KE = \frac{1}{2}\ m\ v_T{}^2$$
$$KE = \frac{1}{2}\,(I/r^2)(r\,\omega)^2$$
$$KE = \frac{1}{2}\ I\ \omega^2$$

A rotating non-point object (like the second ball above) can be analyzed as if it was a combination of millions of tiny masses. If "calculus" is used to find $KE = \frac{1}{2}mv^2$ for each tiny mass and then add them together, the result is: $KE_{object} = \frac{1}{2} I_{object}\,\omega^2$.

For any object, TOTAL KE = translational KE + rotational KE = $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$.

To determine whether both motions contribute to an object's KE, ask "After one complete revolution, has the object moved to a different point in space?". For the rolling sphere, the answer is YES because it has moved forward a distance of $2\pi r$: the sphere has KE_{transl} and KE_{rotn}. For the rock-on-a-string, the answer is NO because it has returned to its starting place: the rock has only one kind of KE (even though this KE can be calculated in two different ways, as shown above).

PROBLEM 5-F: What is the friction direction if a ball ($r = .75$m, $v = 0$, $\omega = 4.0$ rads/s) makes contact with a friction-producing surface? If $v = 3.0$ m/s and $\omega = 4.0$ rads/s? If $v = 3.0$ m/s and $\omega = 0$? If it rolls (without slipping) down a ramp? Up a ramp? In each case, is friction static or kinetic? {The answers are given soon.}

PROBLEM 5-G: Using the Total Work Equation for an "Object Race".

Three 1.50 kg objects (a block, solid sphere and solid cylinder) are at rest at the top of a 3.00 m high, 36.9° ramp. μ_k between the block & ramp is .20, and there is enough friction to make the sphere and cylinder roll without slipping.

Use the Total Work Equation to answer these questions: Which object reaches the bottom of the ramp first? At the bottom, what is the KE of each object?

SOLUTION 5-G

In the rolling-object TWE below, why is $W_{friction} = 0$? {The answers is given soon.}

Sliding-Object T.W.E. (for block)

$$\frac{1}{2}mv_i^2 + mgh_i = \frac{1}{2}mv_f^2 + mgh_f + |\,\mu_k\ N\ d\,|$$
$$0 + m(9.8)(+3) = \frac{1}{2}mv_f^2 + 0 + [.20\,m(9.8)\cos 36.9°]\left(\frac{3.0}{\sin 36.9°}\right)$$
$$6.57\ \text{m/s} = v_f$$

Rolling-Object T.W.E. (for cylinder or sphere)

$$\frac{1}{2}mv_i^2 + mgh_i = \left\{\frac{1}{2}mv_f^2 + \frac{1}{2}\ I\ \omega^2\right\} + mgh_f + |W_{friction}|$$
$$0 + m(9.8)(+3) = \frac{1}{2}mv_f^2 + \frac{1}{2}(Xmr^2)\left(\frac{v_T^2}{r^2}\right) + 0 + 0$$
$$\underbrace{3\cancel{m}g = \frac{1}{2}\cancel{m}v_f^2 + X\left(\frac{1}{2}\cancel{m}v_f^2\right)}$$

For sphere, X=.4 and v_f = 6.48 m/s.

For cylinder, X=.5 and v_f = 6.26 m/s.

Each object has $v_i = 0$, with constant F (and thus constant a) throughout the race. By solving tvvax equations (or using common sense), it can be shown that the object with the largest v_f also has the smallest Δt. The block wins the race.

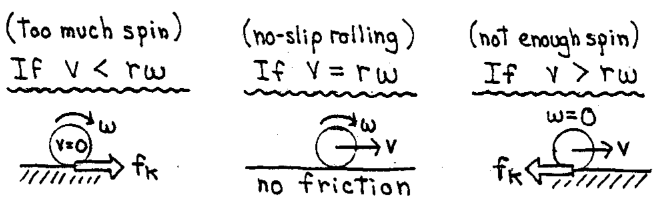
In the equations above, "m" cancels; as in other "Great Races", mass doesn't matter. But "shape" does make a difference. Do you see why the sphere (with X = .4) beats the cylinder (with X = .5)?

The block's final KE is $\frac{1}{2}mv_f^2 = .5(1.5)(6.57)^2 = 32.4$ Joules. The sphere's KE_f is $\frac{1}{2}mv_f^2 + \frac{1}{2}Xmv_f^2 = .5(1.5)(6.48)^2 + .5(1.5)(.4)(6.48)^2 = 31.5 + 12.6 = 44.1$ J. Similarly, the cylinder's KE_f is 29.4 + 14.7 = 44.1 J. Compare these KE's (32.4, 44.1 & 44.1) with $\Delta PE = mg\,\Delta h = 1.5(9.8)(-3) = -44.1$ Joules. Why does the block have only 32.4 J?

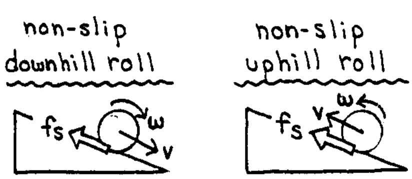
Even though the block has the largest v_f and wins the race, it has the least KE at the end because sliding friction has wasted 11.7 J of KE: $W_{friction} = \mu_k\, mg\cos\theta\, d = .20(1.5)(9.8)(\cos 36.9°)(3/\sin 36.9°) = 11.7$ J. But the friction acting on the cylinder and sphere is static; instead of wasting KE, this "rolling static friction" just changes it from $KE_{translation}$ to $KE_{rotation}$. For example, the sphere's $KE_{rotation}$ of 12.6 J is taken from $KE_{translation}$, decreasing $KE_{translation}$ from 44.1 J (which it would be if there was no friction to produce rolling) to 31.5 J.

{In Problem 5-##, this "Object Race" is solved using F=ma, $\tau = I\alpha$ and tvvax.}

SOLUTION 5-F: The first 3 situations are like a car that 1) over-accelerates at the start of a race (tire rotation is too fast for the car's forward speed, so tires break loose and squeal), 2) rolls smoothly for a while, and 3) over-brakes after the finish line (rotation is too slow for the forward speed, so tires skid). {The logical reasons for these friction directions are discussed in Problems 3-E and 3-25.}



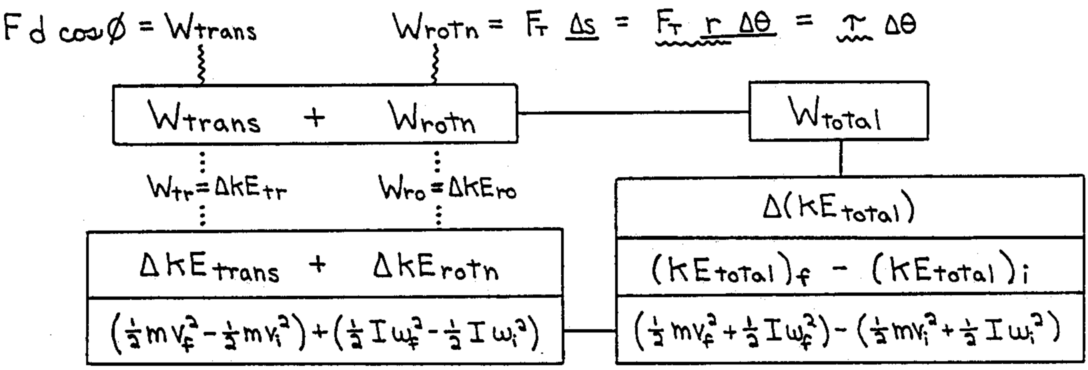
Now think about what kind of sliding will occur if friction = 0 during a downhill or uphill roll, and what direction f_s must be to prevent this "would-be" sliding. Then compare your answers with the diagrams below. What will each f_s do to v and ω?



"Downhill f_s" decreases v (because f_s & v are in opposite directions) and increases ω (because f_s is in the same direction as ω). This is, of course, the same conclusion that was reached in Solution 5-G's KE analysis. During an uphill roll, f_s acts to decrease ω and increase v. How can friction increase v? This is discussed [using mg(sin θ), energy conservation logic, ...] in Problem 5-#.

The main theme of Chapter 4A is "W_{total} produces $\Delta(KE_{total})$". W_{total} can be split into $W_{translation}$ [$W_{trans} = F_{total}\, d\cos\phi$] and $W_{rotation}$ [$W_{rotn} = F_T\,\Delta s = F_T\, r\Delta\theta = \tau\,\Delta\theta$]. Similarly, ΔKE_{total} can be split into $\Delta KE_{translation}$ and $\Delta KE_{rotation}$.

MANY-SIDED EQUATIONS: As in Section 4.7, every box in the diagram below equals every other box. The ⋮ lines show that equations like "$W_{translation} = \Delta KE_{translation}$" and "$W_{rotation} = \Delta KE_{rotation}$" are valid; in Problem 5-##, these equations are used to explore energy conservation for the sphere of Problem 5-G. **Comments: 1) "$F_T\ r\ \Delta\theta$" can be grouped in two different ways, to get $F_T\ \Delta s$ or $\tau\ \Delta\theta$. 2) Compare the 4 terms ($\frac{1}{2} mv_f^2$, $\frac{1}{2} mv_f^2$, ...) in each of the two lowest boxes. Do you see why these two boxes are equal?**



OPTIONAL: If your class studies "rotational power", these formulas will be useful.

$\Delta KE_{trans}/\Delta t = W_{trans}/\Delta t \equiv P_{translation} = F\, v \cos\phi$, and

$\Delta KE_{rotn} = W_{rotn}/\Delta t \equiv P_{rotation} = F_T\, v_T = (\tau/r)(r\omega) = \tau\,\omega$.

$(P_{trans} + P_{rotn})\, \Delta t = P_{total}\, \Delta t = W_{total}$ = every other box in the diagram above.

Part 5: Angular Impulse-Momentum

If both sides of Chapter 4B's "$F_{ext}\, \Delta t = (mv)_f - (mv)_i$" are multiplied by r, and substitutions of $F_{ext} = \tau_{ext}/r$, $m = I/r^2$ and $v = r\omega$ are made, the result is

$$\begin{aligned}
F_{ext}\quad \Delta t &= (m\ v)_f - (m\ v)_i\\
F_{ext}\ r\ \Delta t &= (m\ v\ r)_f - (m\ v\ r)_i\\
(\tau_{ext}/r)\ r\ \Delta t &= [(I/r^2)(r\omega)r]_f - [(I/r^2)(r\omega)r]_i\\
\tau_{ext}\quad \Delta t &= (I\ \omega)_f - (I\ \omega)_i
\end{aligned}$$

"mvr" is given the name *angular momentum*, abbreviated "L". L is calculated like torque (in Chapter 5E), except "mv" is substituted for "F". Here is a review of the steps from Chapter 5E, adapted for calculating L: 1) Draw the object's v-vector. 2) Choose an L-axis. 3) Use the formula $\underline{L = mvr(\sin\theta)}$ where r is the axis-to-object distance and θ is the angle between v and r, or use $\underline{L = mvr_\perp}$ where $r_\perp$ is the shortest distance between the v-extension and L-axis, or use $\underline{L = I\omega}$ because (as derived above) $L = mvr = I\omega$. 4) L-direction (or ± sign) is found the same way as τ-direction. 5) For a system of several objects, L_{total} = the sum of individual L's.

Internal & external torque are caused by internal & external force, respectively. Always ask the question from Section 4.8: "**Is the cause of a force (and the torque it produces) within the system or outside the system?**". Some situations with *internal torque* are collisions, internal T's or N's or friction, a "within-the-system throw" (these are all analogies to the *internal force* situations of Section 4.8), or an ice-skater who moves her arms from outstretched to close-to-her-body.

CONSERVATION OF ANGULAR MOMENTUM: When $\tau_{external} = 0$, $L_i = L_f$.
ALMOST-CONSERVATION: If $\tau_{ext} \neq 0$, but a very small Δt causes $\tau_{ext}\, \Delta t \approx 0$, $L_i \approx L_f$.

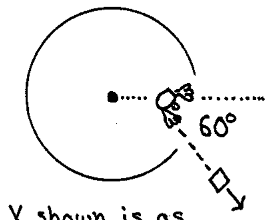
PROBLEM 5-H: Angular Momentum and More

A 100 kg man holding an 8.0 kg box stands .50 m from the outer edge of a motionless 500 kg, 3.00 m radius solid cylindrical disk. The disk rotates around a vertical frictionless axle through its center.

The outer edge of the disk is pushed with a constant counter-clockwise tangential force of 600*N* for 4.00 s; find ω_f of the box.

If you have studied "Angular Work-Energy", find the revolutions made by the disk during this 4 s.

After the 600*N* force stops (at the end of 4 s), the man throws the box with speed 12.0 m/s in the direction shown at the right, then walks 1.50 m toward the center of the disk. Find the disk's ω just after the throw, and its ω after the walk.



V shown is as reported by an observer on ground.

SOLUTION 5-H

Let's define box-man-disk as a system. The disk has "X = .5". The system's initial I_{total} is $I_{box} + I_{man} + I_{disk} = 8(2.5)^2 + 100(2.5)^2 + .5(500)(3)^2 = 2925\ \text{kg m}^2$.

We'll also define 4 "special times": t_1 (before the 4s push), t_2 (after pushing, before box-throw), t_3 (after throw, before walk), t_4 (after walk).

The 800*N* push is $F_{external}$; it causes ΔL and ΔKE during the first 4 s.

$$F_{ext}\ r_\perp\ \Delta t = I\omega_f - I\omega_i$$
$$600(3)(4) = 2925\omega_f - 2925(0)$$
$$2.46\ \tfrac{\text{rads}}{\text{s}} = \omega_f$$

$$W_{rotation} = \Delta KE_{rotation}$$
$$F_T\ r_\perp\ \Delta\theta = \tfrac{1}{2} I\ \omega_f^2 - KE_i$$
$$600(3)\Delta\theta = \tfrac{1}{2}(2295)(2.46)^2 - 0$$
$$\Delta\theta = 4.92\ \text{rads}\ \{\text{or } .782\ \text{rev}\}$$

The box-throw and walk involve only internal forces, so angular momentum is conserved. After the first 4.00 s, the system's total L (which is $L_{disk} + L_{man} + L_{box}$) remains the same: $L_2 = L_3$, $L_3 = L_4$.

To find L-direction (and ± sign), use this analogy to τ-direction; hold the r-vector down at the axis and push it with v (not F, as was done for τ). v_{disk} and v_{man} will rotate the r-vector ↺, which I'm defining to be +. But v_{box} rotates it ↻, which makes the box's L have a – sign. A useful shortcut: if the around-the-axis v (or ω) is ↺, L is +; if it is ↻, L is –.

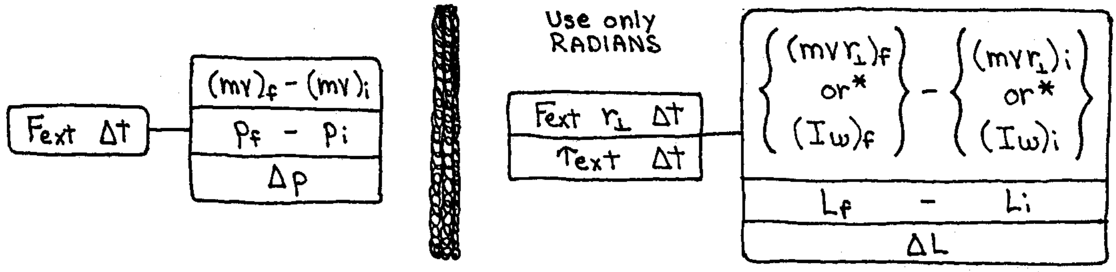
From before-throw to just-after-throw

$$(L_{d+m+b})_2 = (L_{disk})_3 + (L_{man})_3 + (L_{box})_3$$
$$(I_{sys})(\omega_2) = I_{disk}\,\omega_3 + I_{man}\,\omega_3 - mv\,r\sin\theta$$
$$(2925)(2.46) = (2250)\omega_3 + (100)(2.5)^2\omega_3 - [8(12)2.5\sin 60^\circ]^*$$
$$+7195 + 208 = 2875\,\omega_3$$
$$2.57\ \text{rads/s} = \omega_3$$

From before-walk to after-walk

$$(L_{disk})_3 + (L_{man})_3 + \cancel{(L_{box})_3} = (L_d)_4 + (L_m)_4 + \cancel{(L_b)_4}$$
$$2250(2.57) + 100(2.5)^2(2.57) = (2250)\omega_4 + 100(1)^2\omega_4$$
$$3.14\ \text{rads/s} = \omega_4$$

Linear & angular momentum cannot be mixed together. Below, all right-side angular motion boxes equal each other (as in Section 4.7), but they are not equal to the left-side linear motion boxes. This is emphasized by the line separating them.



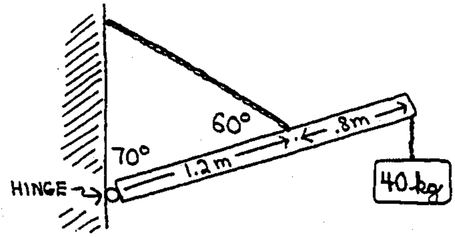
"$m v r_\perp$ <u>or</u> $I\omega$" means that you can calculate L using the most convenient formula. If an object has straight-line motion (like the thrown box), use $m v r_\perp$ [$= mv(r \sin\theta)$]. If a point-object moves in a circle (like the man), use either $m v r_\perp$ or $I\omega$. For a "large object" (like the disk) that has $I = X m r^2$, use $I\omega$.

Optional: The vector properties of angular momentum, $\mathbf{r}\,\mathbf{F} = \tau$, $\mathbf{r}\,\mathbf{p} = \mathbf{L}$ and $\tau\,\Delta t = \Delta L$ are discussed in Section 5.94, using spinning tops and bicycle wheels as examples.

5G: Equilibrium (Torque Statics)

Torque can be used to analyze a situation in which an object remains motionless even though several forces act on it.

PROBLEM 5-I: Torque Statics.
In the picture at the right, the 10 kg board ( ) is symmetric. Find the rope tension, and the vertical & horizontal forces the hinge exerts on the board.



Here is a general strategy for solving torque-statics problems. As you study it, use Problem 5-I to practice what you're learning.

Learn the torque-calculation methods of Chapter 5E: $\tau = F r \sin\theta$ and $\tau = F r_\perp$.

Read the problem, draw a picture, choose an object and draw a force-diagram; you must draw each force at the place where it is actually applied to the object.

The object remains at rest, so v is always zero and all accelerations are 0: $a_x = 0$, $a_y = 0$, $\alpha = 0$. $F_x = ma_x$, $F_y = ma_y$ and $\tau = I\alpha$ simplify to give $\mathbf{F_x = 0}$, $\mathbf{F_y = 0}$, and $\boldsymbol{\tau = 0}$.

Define x & y axes, and substitute all x-forces and y-forces into $F_x = 0$ and $F_y = 0$.

Choose a useful τ-axis*, and substitute all τ's into $\tau = 0$. *When an object rotates, the rotation axis (a door's hinge, the center of a wheel,...) is usually the best choice for a τ-axis. But for a static object, you can choose a τ-axis to be anywhere, so there are an infinite number of τ-axes and corresponding τ-equations. Some axis-choices are better than others. If a force (or its extension) passes through a τ-axis to make $r_\perp = 0$, the F doesn't produce any τ; the F disappears from the τ=0 equation for that particular τ-axis. If you choose a τ-axis wisely, you may eliminate enough F's to get a 1-unknown τ-equation; this is often the key to solving a statics problem quickly.

If you can't get a 1-unknown τ-equation, substitute into F_x=0 and F_y=0 first; they're easier than τ=0. Then check for unknowns and choose a τ-axis accordingly. { Problem 5-# shows how to use a "multiple unknown" algebra strategy. }

For any statics-equation*, you can treat an object as if all of its mass is at the object's center-of-mass. *You cannot do this if an equation contains r^2 instead of r.

SOLUTION 5-I

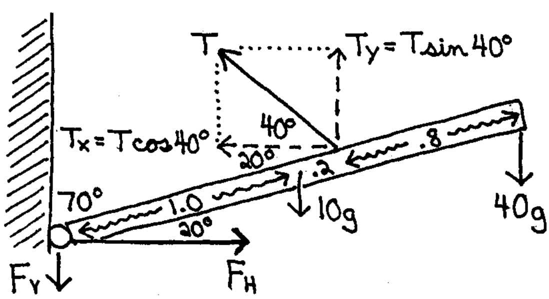
Just follow the steps outlined above, using the board as "object". I've chosen the hinge as τ-axis, because this makes F_v and F_h disappear to give a 1-unknown equation. Solve-and-use links: solve for T, then use it in the ⇓ substitutions.

Draw "10g" at the board's center-of-mass, as if all mass was located there.

To get the 20° angles below and above the board, use two of Section 1.3's "ΔXYZ" tools: Y (angles that add up to 90°), and Z (parallel lines cut by a diagonal).

Compare the angles in the drawing and in the equations. Do you see why 40° is used for T_x and T_y, while θ is 60° in the "τ = F r sinθ" for rope tension?

For substitution into τ=0, I'll be using "F r sinθ" instead of "r F sinθ".



$$F_x = m a_x^{\,0}$$
$$+F_H - T\cos 40° = 0$$
$$\Downarrow$$
$$F_H - (798)\cos 40° = 0$$
$$F_H = 611 \text{ Newtons}$$

$$F_y = m a_y^{\,0}$$
$$-F_V + T\sin 40° - 10g - 40g = 0$$
$$\Downarrow$$
$$-F_V + (798)\sin 40° - 490 = 0$$
$$+23\ N = F_V$$

$$\tau_{total} = I\,\alpha^{0}$$
$$0 + 0 - 10g(1.0)\sin 70° + T(1.2)\sin 60° - 40g(2.0)\sin 70° = 0$$
$$1.039\ T = +92.1 + 737$$
$$T = 798 \text{ Newtons}$$

On the F-diagram, we drew F_v in the ↓ direction. This is opposite the +y direction, so "$-F_v$" was substituted into $F_y = 0$. The + sign in "$+23 = F_v$" does not mean that F_v points in the +y direction. Instead, it says "Yes, your substitution-decision about the F_v direction (that it points downward) is correct; F_v does point downward." { This is discussed in "The Meaning of ± Signs", Section 3.6.}

I recommend the equation-layout shown above. Put F_x=0 and F_y=0 side by side, leave room for some algebra-work. τ=0 is usually longer, so put it on a separate line. { If there are only a few F's, or if you can write small, put all 3 equations on 1 line. }

Be disciplined when you substitute into the τ-equation. For each force on the F-diagram, there should be a term for the τ it produces. If you use "$\tau = \pm Fr \sin\theta$", you have 4 decisions to make for each F: decide the ± sign, the magnitude of F & r*, and the angle between them. { * As I'm writing, I say to myself "10g at 1.0 ..., T at 1.2 ...". } Look at the τ-equation above. Do you see the 4 decisions for each non-zero τ term?

For some problems, including some in Section 5.91, it is easier to use "$\tau = \pm F\, r_\perp$", and instead of 4 decisions there are 3: ± sign, F, and $r_\perp$.

If you want, use other τ-equations to check the answers of T = 798, F_h = 611, F_v = 23↓.
If they're correct, all equations should have left-side = right-side. Do they?
For example, if the rope-attachment point is chosen as τ-axis,

$$+23(1.2)\sin 70° + 611(1.2)\sin 20° + 10g(.2)\sin 70° + 0 - 40g(.8)\sin 70° = 0.$$

$$-295.1 + 294.7 = 0\ ?$$ {This is close enough; round-offs cause some error.}

Chapter 5 Flash-Card Review

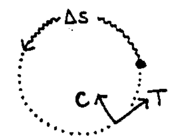
5A	At any instant of time, the C & T axes are __ to each other; C points __ , and T points __ .	perpendicular , toward the circle-center, along motion-line (out front or rear window)
5B	a_c points __ (__ to v_T) and causes __ motion.	toward the circle-center, ⊥ , curving
5B	If F_c points __ the center, its ± sign is __ .	toward (+), away from (–)
5B	If an object changes height (as in __), use __ . If v_T gradually ↓, the object __ first at __ .	a "vertical circle", the T.W.E. ($F\Delta x = \Delta KE$) departs from a circular path, the circle-top
5B	"centripetal" is a __ , not the name for a __ . F is caused by __ , but not by __ or __ . Specifically, F_c can be caused by __ .	direction, new kind of force real objects, "circular motion", acceleration T, N, mg or GMm/r^2, air-lift, friction, ...
5B	If an object moves on a non-circular curve, __ .	use radius-of-curvature for circle-center & r
5C	$F_{gravity}$ magnitude & direction is __ (general) and __ (near earth's surface), so g ≈ __ .	GMm/r^2 center-to-center attraction mg "down" toward earth-center ; GM_{earth}/r^2
5C	$F_{gravity}$ extends to __ , can cause __ of __ .	outer space, orbiting, planets/moons/satellites
5D	A "visual" way to understand a radian is __ .	cut r-string, stretch on rim, draw lines to i & f
5D	3 kinds of motion variables are __ , 2 equation types are __ and __ (units __).	linear, tangential, angular defining, connecting (must be in radians)

5D	All points on a spinning plate have __ but __ .	the same ω, different v_T (if r's differ)
5D	A rolling object (radius R) or the rope on a rotating pulley (radius R) will have __ if __ .	$\Delta x = R\,\Delta\theta$, $v = R\,\omega$, $a = R\,\alpha$ there is no "slipping"
5D	If you know __ of __ (2 answers) you can find the other(s) by using __ and/or __ .	1 of (ω, f, rpm, T); 2 of (v_T, r, ω/f/rpm/T) conversion factors, formulas
5D	For __ motion, __ acceleration is used in equations. {3 answers}	linear (a), constant-speed circular (a_c), changing-speed circular (a_c , a_T, α)
5D	a_T and a_c show the rate-of-change of __ .	v_T magnitude (for a_T) and v_T direction (for a_c)
5D	There is really __ kind of non-angular **a**, __ .	just one, **a ≡ Δv-vector / Δt**
5D	To get $t\omega\omega\alpha\theta$, replace __ , even though __ .	tvax with $t\omega\alpha\theta$, they aren't equal
5E	To find τ , draw each F at __ , choose __ , then use a τ-formula, either __ or __ , where **r** is __ , and $r_\perp$ is __ .	the F-application point , a τ-axis $r\,F\,\sin\theta$, $F\,r_\perp$ a vector drawn from τ-axis to F-point the closest distance from F-extension to τ-axis
5E	To find the ± sign of τ direction, __ . OPTIONAL: To find τ vector direction, __ .	point, push or pull, decide (usually $\circlearrowleft$ is +) curled r-fingers are rotation, r-thumb is τ
5F	For tvvax, use __ units; for other equations, __ .	consistent, you must use radians
5F	I_{total} = __ . Two object types are __ (I = __).	sum of I's, point (mr^2), "large" (X mr^2)
5F	For __ equation, you should know the __ and what __ to look for, learn links by knowing __ , use __ to learn the __ for each variable-letter.	every, if-then requirements for its use, choosing-clues, all **a**-eqns & α-eqns & (etc.) active review, word(s) & units & "look"

You must know the equations! To learn them, study their logical organization in the **Chapter 5 Summary** (it's very good) and use active review to learn their <u>if/then requirement</u>, <u>choice clues</u>, <u>link possibilities</u> and <u>letter meanings</u>.

5F	If a pulley has m≠0 and α≠0, __ ≠ __ . The "link" between F=ma and τ=Iα is __ .	$T_1 \neq T_2$, and τ_1 (= T_1 R) ≠ τ_2 (= T_2 R) $\underline{a} = (a_T)_{rim} = r_{rim}\,\underline{\alpha}$
5F	An object's v is →; f_k points → if __ , ← if __ .	$r\omega > v$ (over-spin), $r\omega < v$ (under-spin)
5F	An object has __ KE if it is __ after 1 rev. An object has __ KE if it is __ after 1 rev.	only one kind of, at the same location both rotnl & transl, at a different location
5F	To calculate L, __ but replace __ .	use τ-calculation strategy, F with mv
5F	To determine whether τ is "external", ask __ . Some situations with $\tau_{internal}$ are __ .	Is the τ-causer a part of the system? $F_{internal}$ analogies, ice skater's arm-changes
5F	Conservation of L occurs if __ .	$\tau_{ext} = 0$; $\Delta L \approx 0$ if $\Delta t \approx 0$ causes $\tau_{ext}\,\Delta t \approx 0$
5F	__ can be added, but __ cannot be "mixed".	KE_{rotn} & KE_{trans}, linear & angular momentum
5F	L = __ for linear v, rock-on-string, large object.	$mvr_\perp$, $mvr_\perp$ or $I\omega$, $I\omega$
5G	If an object is static, __ , __ and __ . A force vanishes from a τ-equation if __ . Make __ τ-decisions for Fr sinθ, __ for $Fr_\perp$.	$F_x = 0$, $F_y = 0$, $\tau = 0$ for every τ-axis F-extension passes thru the τ-axis (so $\tau = 0$) 4 (±, F, r, sinθ), 3 (±, F, $r_\perp$)

Chapter 5 Summary



The centripetal (radial) axis points toward the circle's center, along a radius-line. The tangential axis points along the direction of motion (straight out the "front windshield" or "rear window").

At any instant of time, the centripetal and tangential axis-directions are perpendicular ($\perp$) to teach other.

A car's speedometer shows v (for straight-line motion) or v_T (if motion is along a curve). Δs is the distance an object actually travels (see picture above).

Centripetal (Radial) Acceleration: At any instant of time, whether v_T is constant or changing, a_c is $\perp$ to v_T and points toward the circle-center, with magnitude $v_T{}^2/r$.

Cause $\rightarrow$ Effect: A force "F_c" causes "m" to move in a circle with acceleration "a_c",

$$F_c = m\,\frac{v_T{}^2}{r} \quad \text{—(by substituting "} v_T = r\omega \text{" from 5C)}\!\!\rightarrow \quad F_c = m\,r\,\omega^2$$

F_c toward center is +, F_c away from center is –; the F_T component doesn't cause a_c.

"Centripetal" is a direction (like "x" or "y"), not the name for a new kind of force.

F_c is caused by real objects; "circular motion" or "acceleration" don't cause force.

$F_{gravity}$ = $\mathbf{GMm/r^2}$, center-to-center attraction; G = 6.67 x 10^{-11} (in SI units).

Near the earth's surface, $F_{gravity}$ = mg, straight down toward earth's center; $g \approx GM_{earth}\,m/r^2$.

$F_{gravity}$ extends into "space"; it can cause one object to *orbit* around another object,

$$\frac{GMm}{r^2} = m\frac{v_T{}^2}{r} \qquad \frac{GM}{4\pi^2}T^2 = r^3 \qquad GM = r^3\omega^2$$

Angular-Motion Definitions, and Connecting Equations

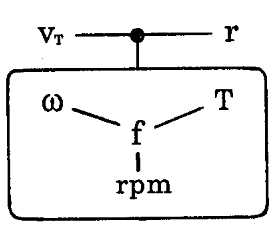
$$\Delta s = r\,\Delta\theta$$

$$\frac{\Delta s}{\Delta t} = v_T \;\text{——}\; v_T = r\,\omega \;\text{——}\; \omega \equiv \frac{\Delta\theta}{\Delta t}$$

$$\frac{\Delta v_T}{\Delta t} = a_T \;\text{——}\; a_T = r\,\alpha \;\text{——}\; \alpha \equiv \frac{\Delta\omega}{\Delta t}$$

Angular Velocity Units

If you know 1 of 4
(ω , f , rpm , T)
you can find the others:
2π rads = 1 rev = 360°,
60 seconds = 1 minute,
$\omega = 2\pi f$; $1/f = T$, $f = 1/T$.



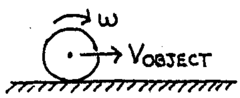
If you know 1 of 3
(v_T , r , ω/f/rpm/T)
you can find the others:
$v_T = r\,\omega$
$v_T = 2\pi r/T$

FOUR KINDS OF ACCELERATION

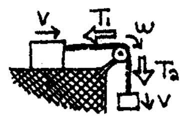
	a LINEAR	a_T TANGENTIAL	α ANGULAR	a_c (or a_R) CENTRIPETAL
SITUATIONS: LINEAR MOTION	a is "REGULAR" ACCELERATION (as in Chapters 2-4)	not used	not used	0
SITUATIONS: CONSTANT-SPEED CIRCULAR MOTION	not used	0	0	a_c shows the rate-of-change of $\vec{v}$-direction as object moves along a curve.
SITUATIONS: CHANGING-SPEED CIRCULAR MOTION	not used	a_T shows v_T's rate-of-change	α shows ω's rate-of-change	
F = ma EQUATION	$F_x = m a_x$, $F_y = m a_y$	$F_T = m a_T$	$\tau = I \alpha$	$F_c = m a_c$
MAGNITUDE	$\frac{\Delta v}{\Delta t}$	$\frac{\Delta v_T}{\Delta t}$; $a_T = r \alpha$	$\frac{\Delta \omega}{\Delta t}$; $a_T = r \alpha$	v_T^2/r and $r\omega^2$
DIRECTION	If speed ↑, out "front window". If speed ↓, out "rear window". {Along direction of motion.}	If speed ↑, out "front window". If speed ↓, out "rear window". {Along direction of motion.}	α has same ± sign as a_T. {see * below}	toward center of circle (on radial-line)
What is changing?	Magnitude of v	Magn. of v_T	Magn. of ω	Direction of v vector

There is one kind of non-angular acceleration: **a-vector** ≡ **Δ(v-vector)** / Δt. **a**, $\mathbf{a_c}$ & $\mathbf{a_T}$ are just convenient categories that describe the Δv / Δt for three common situations. To get the circular-motion $\mathbf{a}_{total}$ vector, add $\mathbf{a_c}$ and $\mathbf{a_T}$ (which are always ⊥) as vectors

COMBINED MOTION: TRANSLATION + ROTATION



$(\Delta x)_{object} =^* (\Delta s)_{rim} = r_{rim}\,\Delta\theta$
$v_{object} =^* (v_T)_{rim} = r_{rim}\,\omega$
$a_{object} =^* (a_T)_{rim} = r_{rim}\,\alpha$

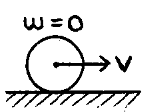


$(\Delta x)_{blocks} = (\Delta x)_{rope} =^* (\Delta s)_{rim} = r_{rim}\,\Delta\theta$
$v_{blocks} = v_{rope} =^* (v_T)_{rim} = r_{rim}\,\omega$
$a_{blocks} = a_{rope} =^* (a_T)_{rim} = r_{rim}\,\alpha$

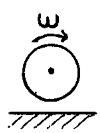
* These ='s are true only if the object (or rope) moves across the floor (or pulley) without slipping.

All points on a spinning plate have the same ω;
but if two points have different r's, they will (because $v_T = r\omega$) have different v_T's.

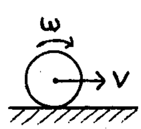
KINETIC ENERGY (translational & rotational)



only KE_{tran}, $\frac{1}{2} m v^2$



only KE_{rotn}, $\frac{1}{2} I \omega^2$



$KE_{tran} + KE_{rotn}$, $\frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$

If object is at same point after 1 rev (like rock-on-string), it has one kind of KE; this can be calculated as $\frac{1}{2}mv^2$ or $\frac{1}{2}I\omega^2$. If object has moved after 1 rev (like a rolling sphere), it has KE_{trans} & KE_{rotn}, and if rolling is non-slip so $v_T = r\omega$, $KE_{total} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + X(\frac{1}{2}mv^2)$.

Linear, Tangential & Angular Variables, and CONNECTING EQUATIONS

LINEAR	t	Δx	v	a	F	m
TANGENTIAL	t	Δs	v_T	a_T	F_T	m
ANGULAR	t	$\Delta\theta$	ω	α	τ	I
CONNECTING EQUATIONS	—	$\Delta s = r\,\Delta\theta$	$v_T = r\omega$	$a_T = r\alpha$	$F_T = \frac{\tau}{r}$	$m = \frac{I}{r^2}$

Linear, Tangential and Angular Equations

Linear equations (like $F = ma$) have "tangential analogies" ($F_T = m\,a_T$). If *connecting equation substitutions* are made for each tangential variable ($F_T = \tau/r$, $m = I/r^2$, $a_T = r\alpha$), every "r" will cancel, as shown in Parts 1, 3 & 4 of Chapter 5F. The overall result is that linear variables change to tangential and then angular, even though (as discussed in 5D Part 4) tangential & angular variables are not equal.

LINEAR EQUATIONS

tvvax (if a is constant)

$$v_f - v_i = a t$$
$$\Delta x = \tfrac{1}{2}(v_i + v_f)\,t$$
$$\Delta x = v_i t + \tfrac{1}{2} a t^2$$
$$\Delta x = v_f t - \tfrac{1}{2} a t^2$$
$$v_f^2 - v_i^2 = 2 a \Delta x$$

Use Chapter 2 strategies.

$\longrightarrow$

TANGENTIAL EQUATIONS

$t\omega\omega\alpha\theta$ (if α is constant)

$$\omega_f - \omega_i = \alpha t$$
$$\Delta\theta = \tfrac{1}{2}(\omega_i + \omega_f)\,t$$
$$\Delta\theta = \omega_i t + \tfrac{1}{2}\alpha t^2$$
$$\Delta\theta = \omega_f t - \tfrac{1}{2}\alpha t^2$$
$$\omega_f^2 - \omega_i^2 = 2\alpha\Delta\theta$$

Use analogies of tvvax strategies.

$$F_{total} = ma \longrightarrow \tau_{total} = I\alpha$$

$F_{ext}\,\Delta t$ — Δp; $p_f - p_i$; $(mv)_f - (mv)_i$

LINEAR & ANGULAR MOMENTUM cannot be "mixed"

$\longrightarrow$

$F_{ext}\, r_\perp\, \Delta t$; $\tau_{ext}\,\Delta t$ — ΔL; $L_f - L_i$; $\begin{pmatrix} I\omega \text{ or} \\ mvr_\perp \end{pmatrix}_f - \begin{pmatrix} I\omega \text{ or} \\ mvr_\perp \end{pmatrix}_i$

$F d \cos\phi$; $F_x\,\Delta x$ ⋮ W_{trans}

$F r_\perp \Delta\theta$; $\tau\,\Delta\theta$ ⋮ W_{rotn}

$W_{trans} + W_{rotn}$ — W_{total}

OPTIONAL: $(F v\cos\phi + \tau\omega)\,\Delta t$; $P_{total}\,\Delta t$ — W_{total}

$W_{tr} = \Delta KE_{tr}$ | $W_{ro} = \Delta KE_{ro}$

$$\Delta KE_{trans} + \Delta KE_{rotn}$$
$$\left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + \left(\tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2\right)$$

W_{total} —

$$\Delta(KE_{total})$$
$$(KE_{total})_f - (KE_{total})_i$$
$$\left(\tfrac{1}{2}mv_f^2 + \tfrac{1}{2}I\omega_f^2\right) - \left(\tfrac{1}{2}mv_i^2 + \tfrac{1}{2}I\omega_i^2\right)$$

tvvax strategies (as in Section 2.21 or Chapter 2's Summary) can be used for $t\omega\omega\alpha\theta$: read/think/draw, choose i & f points for a constant-α interval, make a $t\omega\omega\alpha\theta$ table, look for 3-of-5, choose a 1-out equation, substitute and solve, answer the question.

UNITS: For $t\omega\omega\alpha\theta$, just be consistent; use all rads-and-s, or all revs-and-s, or...
<u>For other rotational-motion equations</u>, <u>use only radians</u> for $\Delta\theta$, ω and α.

I: MOMENT OF INERTIA calculation

For a system of several objects, I_{total} = sum of I's for the individual objects.
$I = m r^2$ for a point-object, $I = X m r^2$ for a large object (get X-value from a table).
Optional: *parallel-axis* ($I = I_{cm} + mh^2$) & *radius of gyration* ($I = m r_g^2$), Problems 5-## & 5-##.

How to calculate TORQUE, τ

1) Choose object, draw F-diagram with each F acting at F-point (where F is applied).
2) Choose a specific τ-axis (τ is always calculated "with respect to" a specific axis).
3) Use either of the τ-formulas shown below. (I recommend that you learn both formulas.)

$\underline{\tau = \pm r F \sin\theta}$	$\tau = \pm r_\perp F$
r is a vector from τ-axis to F-point θ is angle between r and F	To find $r_\perp$, a) DRAW the F-extensions, b) find closest approach to τ-axis (at 90°); c) this extension-to-axis distance is $r_\perp$.

4) To find direction-sign of τ, a) POINT pen in r-direction, b) HOLD pen at τ-axis, c) PUSH/PULL pen with F at F-point, d) DECIDE (usually is defined to be +).
5) τ_{total} = sum of individual τ's.

ANGULAR MOMENTUM (L) is calculated almost like τ; just substitute mv for **F**.
r is vector from object-location to L-axis, $r_\perp$ is shortest shortest distance from v-extension to L-axis.
Linear motion: $L = m v r_\perp$. Rock-on-string: $m v r_\perp$ or $I\omega$. "$I = X m r^2$" object: $L = I\omega$.

Separate τ into τ_{int} and τ_{ext}; ask "Is the τ-causer inside or outside the system?". Some examples of τ_{int} are "analogies to $F_{internal}$", and an ice skater's arm-extension.

CONSERVATION OF ANGULAR MOMENTUM: If $\tau_{external} = 0$, $L_i = L_f$.
ALMOST-CONSERVATION: If $\Delta t \approx 0$ causes $\tau_{external}\Delta t \approx 0$, $L_i \approx L_f$.

TORQUE-EQUILIBRIUM PROBLEMS

If an object is "static" (remaining at rest), it has $F_x = 0$ and $F_y = 0$ and $\tau = 0$.
For $\tau = 0$, choose τ-axis anywhere. (If an F-extension goes through a τ-axis (so $r_\perp = 0$), this F disappears from "$\tau = 0$" for that axis. Try to get a 1-unknown equation, or 2-unknowns/2-equations.)

Each F causes a τ; there are 4 τ-decisions for $\tau = \pm F r \sin\theta$, 3 for $\tau = \pm F r_\perp$.