it was not until Georges Cuvier got his hands on these images that the mysteries of this ancient creature began to unravel. In 1796, Cuvier produced a paper documenting the anatomy of this creature, placing it in the family tree of mammals, and finally giving it a name: *Megatherium americanum* (which translates to "great American beast"). Through careful comparative work, Cuvier recognized that this animal was new to science, but clearly related to the edentates, a grouping of mammals that includes armadillos and sloths. This work marked the beginning of Cuvier's prodigious career and helped to provide evidence that the ancient world was full of creatures that are not represented in the modern fauna. Additional fossils of related creatures would be found in later years, and after some further debate, the great anatomist Richard Owen would eventually demonstrate that *Megatherium* was an extinct species of giant ground sloth.

Pimentel uses these two stories to explore many topics along the way. While some digressions are more interesting and germane than others, they generally raise intriguing ideas inspired by the tales of the rhinoceros and *Megatherium*. Pimentel recurrently explores topics such as "the role of imagination in the manufacture of scientific and historical facts" (p. 6), the power of images to convey reality mixed with "preconceptions and mental resonances" (p. 103), and the "alliance between art and science" (p. 164) that gave rise to the discipline of scientific illustration. In telling these tales, he also conveys the importance of understanding how our collective knowledge has changed across centuries. He discusses how the discovery of fossils presented a challenge for many eighteenth-century naturalists, who believed in the doctrine of plenitude and the fixity of species. In so doing, he briefly covers the infancy of paleontology, the debate between uniformitarianism and catastrophism, and the tensions that existed between science and faith during this time, pointing out that religion actually played an important role in the development of earth history and science in general.

If readers are in search of a more systematic and thorough history of paleontology or zoology, then they should look elsewhere. However, Pimentel's extended essay about the "circular biographies" (p. 287) of the rhinoceros and *Megatherium* offers plenty of historical illustrations (56 in total) and rich stories that will inspire further thought about the natural world, how we engage with that which is unfamiliar, and the role of imagination and images in helping us see the reality around us.

*Reviewed by Ryan M. Bebej, Assistant Professor of Biology, Calvin College, Grand Rapids, MI 49546.*

## Mathematics

**A MATHEMATICIAN'S LAMENT: How School Cheats Us Out of Our Most Fascinating and Imaginative Art Form** by Paul Lockhart. New York: Bellevue Literary Press, 2009. 144 pages. Paperback; $14.95. ISBN: 9781934137178.

**MEASUREMENT** by Paul Lockhart. Cambridge, MA: Harvard University Press, 2012. 407 pages, with index. Paperback; $20.50. ISBN: 9780674057555.

**ARITHMETIC** by Paul Lockhart. Cambridge, MA: Harvard University Press, 2017. 223 pages, with index. Hardcover; $22.95. ISBN: 9780674972230.

You will forgive me if I find it normal for mathematics education to be under attack. That has been my experience since the mid-1960s. I wasn't subjected to "new math" in the classroom (we weren't that up-to-date), but I was privileged to attend a National Science Foundation Saturday course aimed at introducing talented high school students in the Chicagoland area to the modern abstract view of mathematics. The short text we used developed the real number system as equivalence classes of Cauchy sequences, claiming this would help us understand what creative mathematics was really all about. I stumbled out of those lectures in a fog of confusion, none the wiser for the honor, yet still interested in mathematics as I understood it.

I underwent the same anxious muddle about three years later during my first semester of abstract algebra, but this time the haze gradually cleared, and I began to appreciate an abstract formal viewpoint. I was not convinced, however, that imposing a set-theoretic foundation on school mathematics was pedagogically or philosophically sound, nor that it would help catapult the USA ahead of the Soviet Union in the space race. Aspects of the New Math reform appealed to me, but I also resonated with parts of Morris Kline's hyperbolic rant *Why Johnny Can't Add: The Failure of the New Math* (1973). The more concrete heuristic approach taken by British mathematics educators under the leadership of Edith Biggs seemed far more promising than what new math proponents had on tap.

Since the 1960s a host of professional documents by committees and individuals have detailed what's wrong with mathematics education in the USA on all levels and have told us what we should do to fix it. Progress has been made on a number of fronts, but not everyone has clambered aboard one of the reform trains. Paul Lockhart, for instance, begs to differ with how things still typically go—actually, he stridently excoriates today's mathematics educators, textbook companies, and conventional schooling.

After finishing a PhD in mathematics, Lockhart taught university mathematics but soon became disillusioned with student attitudes and institutional objectives. He therefore shifted down to the high school level and lower, where he hoped he could instill a love for genuine mathematics before students were corrupted by traditional curricula, mindless worksheets, and uninspiring teachers. In 2002, he penned a 25-page stinging broadside against the status quo in mathematics education, which, after Keith Devlin highlighted it in two 2008 *Devlin's Angle* posts ("one of the best critiques of current K–12 mathematics education I have ever seen"), gained increased notoriety and circulation. Lockhart's 2009 book includes this essay as its opening "Lamentation," concluding with a shorter "Exultation" in which he describes his delight in constructing the mathematical world of the mind, where one's hamsters (a favorite metaphor for mathematical entities) can have all the beautiful functionality anyone would ever want, living in a universe subject only to human imagination and logical consistency.

Lockhart's *Lament* ends by exhibiting some examples of what learning mathematics ought to be like: one problem from number theory, solved using Pythagorean-like arrangements of imaginary rocks (why do successive odd numbers add up to a square?); another from geometry, solved using reflective symmetry (what is the shortest linear path connecting two points via an intermediary point on a straight line?); and a third from combinatorics, tantalizingly left for the reader to solve (must at least two people at a party always have the same number of friends present?). Lockhart's colloquial exposition of these problems and their solutions is clear and engaging. His parting advice to students and teachers is to "throw the stupid curriculum and textbooks out the window" and "just play" with the mathematical creations you dream up (p. 139).

So what would such teaching/learning look like? An extended model of how to pursue real mathematical understanding—of how to explore and discover mathematical connections, using elegant arguments—is implicitly presented in Lockhart's subsequent books, *Measurement* and *Arithmetic*.

Of the two books, *Measurement* is the more ambitious and substantial. The material is divided into two equal parts: the first, Size and Shape (topics in classical and projective geometry, as well as trigonometry); and the second, Time and Space (matters handled by coordinate geometry and differential calculus), in which motion plays an important role in generating curves and sweeping out regions as well as being a concept to analyze mathematically.

After explaining that mathematics is simply an exploration of the perfect patterns of things we create with our minds, to find out how they behave and why, Lockhart offers some problem-solving suggestions: solve problems of your own making; collaborate with others; mess around with ideas even if they seem far-fetched; be open-minded and flexible about whether your conjectures are true; review, critique, and improve your proofs; have fun. Not quite Polya's *How to Solve It* (1945) or his two-volume *Mathematical Discovery* (1962, 1965), but some pointers worth heeding.

It is difficult to summarize the contents of *Measurement* because Lockhart occasionally observes his own advice, to follow a problem to wherever it meanders off. His asides are often stated as observations to be tested or posed as problems for further exploration, a feature that may make the book a good choice for group exploration, though readers are on their own with respect to the answers. But his main topics are organized in an interconnected way around the general theme of the title.

Measurement, he notes, is about comparing one measure with another. As geometry has no natural units (with the exception of a full circle for angles), measurements are intrinsically relative—they are ratios, leading to formulas that relate different measures. Shapes are characterized in terms of similar figures, where one is a scaled version of the other, involving proportional measures. Lockhart also compares lengths, areas, and volumes of a wide variety of figures with one another, giving rise to some nicely argued classic results—Heron's Formula for the area of a triangle; the Pythagorean Theorem and its generalization to the Law of Cosines; areas for a circle and an ellipse; the volumes of a cylinder, pyramid, cone, and sphere; and so on.

Fairly early in the section, Lockhart introduces the so-called classical "method of exhaustion," "by far the most powerful and flexible measuring technique ever devised" (p. 70), as a key strategy for extending results about rectilinear figures to curved ones. A circle, for instance, is approximated ever more closely (gets exhausted) by inscribed regular polygons as their number of sides increases. The polygons' areas tend toward that of the circle, giving the circle's area in the end as half the product of its radius and circumference. A similar idea works for volume comparisons: a cylinder is exhausted by a collection of abutting rectangular boxes, a cone by a stack of cylindrical discs, a pyramid by stacked boxes, a sphere by thin tetrahedra emanating from the center. Using these approximations, Lockhart establishes a number of familiar volume and surface area results known since Euclid and Archimedes. He argues these results informally and concisely, but gives enough details for the reader to follow his reasoning.

Unwilling to admit infinity into mathematics, the Greeks had linked their exhaustion technique to a rigorous double proof by contradiction strategy (a circle's area is neither more than nor less than half its diameter times half its circumference), but this is an idea too complex for *Measurement* to include. Lockhart instead treats the strategy as realized in the infinite limiting process more fully developed in calculus. He also uses the method of exhaustion to argue for the validity of Cavalieri's Principle, which compares lower-dimensional cross sections of figures in order to relate an unknown measure (area, volume) to one that's already known. Lockhart employs this resourcefully in determining the volumes and surface area of a sphere and a torus, the latter result first appearing in a work by Pappus.

Another topic of classical geometry that Lockhart investigates is that of conic sections (first studied by Apollonius), something that has fallen somewhat out of favor in today's streamlined mathematics curriculum. For example, he introduces an ellipse as a dilation of a circle, as a planar projection of a circle, and as a cross section of a cylinder. He then presents an "ingenious argument" using Dandelin spheres for the ellipse's "shockingly beautiful" characterization in terms of foci—"Is that gorgeous, or what!" (p. 145), following this with a discussion of the ellipse's remarkable tangent property—all done without a stitch of algebra or coordinate geometry. The ellipse and other conic sections are then explored using some ideas from projective geometry.

The section on Size and Shape concludes by introducing the helix and the cycloid. As these figures are best understood as traced out by a moving point, Lockhart uses them to segue into the second section of the book, Time and Space. Here he leaves ancient Greek geometry behind to take up seventeenth-century concerns and approaches.

Basic to the modern treatment of shapes is setting up a coordinate system, done to facilitate the use of algebra, including vectors, for analyzing curves. Although at first Lockhart denigrates this—"It's ugly, and should be avoided whenever possible" (p. 214)—he later lauds this way of representing geometric objects, saying that "the connections between algebra and geometry that are revealed by this point of view are among the most fascinating and beautiful results in all of mathematics" (p. 246) and "This viewpoint not only has the benefit of simplicity … but also tremendous flexibility and generality" (p. 295).

Lockhart employs graphed curves to represent and analyze moving points, such as a point on a circle rolling along a line, which produces a cycloid path. Using trigonometric ideas introduced earlier in the book, he determines the parametric equations of the cycloid, later returning to determine its velocity as well as the area and path length for one arch of the curve.

Lockhart adopts a Newtonian view of a curve as traced out by the endpoint of a moving line whose instantaneous velocity $\dot{p}$ is the terminal value of approximating average velocities, attained as time $t$ shrinks to an instant and position $p$ becomes stationary. This is Newton's fluxion, now termed the position's time derivative. After discussing this for motions in more than one dimension, he introduces Leibniz's differential notation $dx$ to denote the instantaneous rate of change of any variable $x$, making $\dot{p} = dp/dt$. Lockhart next develops a collection of formulas for how the d-operator interacts with various arithmetic operations as well as a simple library of formulas for some basic mathematical functions—a plan familiar to anyone who's taught calculus. He then notes that Leibniz's differential calculus can be used to express and solve "virtually all measurement problems" (p. 319), provided these measures are put into motion: "If you want to measure something, wiggle it" so that "it has a rate of motion" (p. 330) one can calculate with.

A "fantastically beautiful and powerful application of the differential calculus [that is] possibly the most useful" (p. 351) is that of optimization. Differentials can be used, for instance, to determine the largest cone that can sit inside a sphere or the precise shape of a cylindrical can that maximizes the amount of soup relative to the amount of metal in the container. The key principle behind these calculations [an early version of which was known to Kepler] is that "when a variable peaks, its differential must vanish … undoubtedly one of the simplest and most powerful discoveries in the history of analysis" (p. 355).

Putting differentiation into reverse, integrals can be calculated to determine areas, volumes, and lengths, provided the formulas are simple enough—though, like almost all invertible procedures, complications can arise even for some familiar curves. This is the case for most arc length calculations, but it even occurs for area calculations. The area under the hyperbola $y = 1/x$ between $x = 1$ and $x = w$, for example, turns out to be complicated, but its properties enable it to be used to define natural logarithms in a rigorous way.

*Measurement* takes us on a rather impressive tour of various fascinating and significant technical results, visiting many high points in geometry and calculus, whose study would be beneficial for prospective middle school and high school mathematics teachers. The text might also be given to a bright and curious student on these levels, but having a guide familiar with the terrain would be advisable. Lockhart provides a superb big picture exposition of the main contours of introductory calculus, but without all the specifics, terminology, and applications present in today's monstrous calculus texts.

Lockhart's goal in *Measurement* was to demonstrate "What a wild and amazing place mathematical reality is! … a vast, ever-expanding jungle … a meeting place for language, pattern, curiosity, and joy" (pp. 396–97). Those of us interested in making mathematics education attractive can only applaud his effort. Keith Devlin goes so far as to say in his Foreword to *A Mathematician's Lament*, "I will tell you this. I would have loved to have had Paul Lockhart as my school mathematics teacher."

*Arithmetic* is the latest book in Lockhart's series, focused, as one would expect, on the most basic aspects of elementary mathematics. We need to count, compare, gather together, remove, multiply, and divide up quantities of things in all parts of our lives and then often record the results. Arithmetic is the art humanity has developed for doing these things in efficient ways. While computation was once a practical skill we needed to hone, Lockhart notes that today's calculators and phones are faster and more accurate than we will ever be, relieving us of its drudgery. However, we can still appreciate and enjoy the underlying ideas and methods of arithmetic as an intellectual craft designed to organize and communicate numerical information, as a sort of "symbol knitting."

As a human construct, arithmetic has a rich and varied history, though this isn't typically explored in mathematics textbooks. Lockhart, however, interweaves his explanations of the main ideas involved in doing different sorts of calculations with occasional accounts of how arithmetic developed in various cultures, both real and imaginary.

While numbers don't mind how they are conceptualized or symbolically represented, such choices do affect how we calculate with them. Lockhart highlights the importance of uniform grouping (adopting a number base) as he discusses the counting systems of three fictitious tribes, tally marks, Egyptian hieroglyphic numerals, Roman numerals, and Chinese named-place-value numerals.

The all-important place-value principle, which makes it possible for us to represent numbers of any size whatsoever, was initially embodied in an abacus, in which different columns or rows stood for different group-levels (one, ten, hundred). We know such artefacts were used for making calculations in many ancient cultures, but the first written place-value system was the Mesopotamian sexagesimal place-value system. Lockhart chooses not to discuss this, only recognizing the Babylonians for using sixty as their rather cumbersome base, but without offering any possible reason for their choice. He instead introduces a written place-value system in the context of discussing our Hindu-Arabic numeration system, which originated in sixth-century India.

Over several chapters, Lockhart reconstructs how the usual algorithms that Europeans eventually adopted for addition, subtraction, multiplication, and division can be based both on the meaning of the operations and on the way we symbolize our numbers. This is done mainly for positive integers, but he notes that it can be extended to calculations involving decimal fractions, whose origin he seems to associate with the French Revolution's proposal to decimalize all measures (the metric system) rather than attributing it to Stevin's landmark treatise two centuries earlier or noting its connection with the much earlier sexagesimal system or Chinese decimal notation or medieval Arabic developments. He also devotes a chapter to discussing how these computational procedures were mechanized over time, from using wheels, gears, and carry pins to electronic circuits and LED displays.

Lockhart concludes his treatment of different number types toward the end of the book by discussing the arithmetic of fractions and negative numbers, inexplicably omitting real and complex numbers. He briefly refers to a couple of historical ways of dealing with fractions (Egyptian) and negative numbers (debts), but much more could have been done along these lines to motivate the ideas and procedures involved, which would connect our understanding of them with how they actually arose. In *A Mathematician's Lament*, Lockhart rued the fact that "we have a mathematics curriculum with no historical perspective or thematic coherence" (p. 56), but *Arithmetic* misses some natural opportunities to remedy this deficiency. For example, China's use of red and black counting rods for signed integers and their rules for calculating with negative numbers in the context of solving linear system problems parallels Lockhart's explanation using sheep and antisheep. Likewise, Arabic and European calculations with subtracted quantities provide a heuristic motivation for multiplying signed numbers. Lockhart's explanations are consistent, however, with his overall perspective on mathematics as a human creation, imaginatively invented. What's most important for him, it seems, is for teachers to reconstruct standard mathematical ideas in ways that charm and entice students to explore them recreationally, even if they involve imaginary hamsters and antisheep rather than practical concerns grounded in historical realities.

Though I very much enjoyed Lockhart's books, I have some reservations and criticisms that go beyond the historical observations just made. These pertain to his basic educational philosophy of mathematics. Lockhart holds that mathematics is ultimately a human mental creation, an art done purely for intellectual enjoyment. He repeats this refrain in a number of contexts, to the point that it gets rather old. Geometry, he insists in *Measurement*, deals with the ideal shapes we define and explore: "none of the things we've been talking about are real … We made up imaginary points, lines, and other shapes so that things could be simple and beautiful—we did it for art's sake" (p. 169). While this seems harder to assert of quantities, which we experience more precisely, he says in *Arithmetic* that he also conceives of numbers as abstract creatures to which we assign behaviors according to our own aesthetic sensibilities (think: negative numbers). Computation has practical applications, but he still claims that "the idea with arithmetic is to have some fun, keep track of a few things, and occasionally enjoy a bit of cleverness" (p. 24). Mathematicians prefer the "purely mathematical realm" for its "sheer intellectual pleasure and entertainment," a universe of exact abstract entities created with "simplicity and abstract beauty" in mind. This may approximate the "fuzzy, random, and inexact" world we live in, but that's not why mathematicians do mathematics (p. 163). Reality provides us with "crude" and "clumsy prosaic object[s]" about which we could never assert any mathematical truths (p. 181). It provides a springboard for humans to create an imaginary world of perfectly behaved objects: "the whole enterprise is a made-up game in our heads" (p. 193).

While I agree that mathematics is not a utilitarian enterprise, this admission does not lead me to ignore its essential connections to a broader reality. A cursory familiarity with the history of mathematics gives the lie to artistic intellectual elitism. Teachers do need to find ways to motivate students to study mathematics, but a practical situation can often do this as well as a game or a whimsical exploration of an idea. Dealing concretely with arithmetic and geometry is important on lower levels, and connecting them with nonmathematical contexts expands students' understanding of the value and interest of mathematical ideas and procedures. Mathematics deals with quantitative, spatial, and kinematic patterns in a given creation already structured by God. Its applicability lies not in humans' brains being part of reality, but in the world being structured as a coherent whole by its Creator. Humans have found ingenious ways to interact mathematically with their everyday contexts, but acknowledging this is quite different from crediting us with creating mathematical reality out of conceptual whole cloth.

Lockhart's antipathy toward real-life applications makes him downplay a side of mathematics that can be helpful to teachers and students. Although I find some of his critique of mathematics education valid, it does not fairly take into account the creative ways some teachers and texts try to connect with students. Lockhart is not alone in wanting to incite a love for mathematics. Regardless, his impassioned advocacy in these books for making mathematics come to life through active explorations of important ideas may inspire such teachers to further improve their own teaching.

*Reviewed by Calvin Jongsma, Professor of Mathematics Emeritus, Dordt College, Sioux Center, IA 51250.*

## ORIGINS

**THE INTELLIGENT DESIGN DEBATE AND THE TEMPTATION OF SCIENTISM** by Erkki Vesa Rope Kojonen. New York: Routledge, Taylor & Francis, 2016. 226 pages. Hardcover; $150.00. ISBN: 9781472472502. eBook; $50.00. ISBN: 9781315556673.

Writing from a theologian's perspective, Erkki Vesa Rope Kojonen argues that "beliefs about the purposiveness or non-purposiveness of nature should not be based merely on science. Rather, the philosophical and theological nature of such questions should be openly acknowledged." He cogently spells out the landscape of the debate over intelligent design, exploring historical approaches to the fundamental question of teleology in nature and showing the importance of the theological and philosophical aspects of design.

Rope Kojonen is a postdoctoral researcher in the Faculty of Theology at the University of Helsinki. His studies and research interests focus on the general discussion between faith and reason with specific