Dr. Marquart: There was an article that came out in the Christian Digest a couple of years ago about this question of imagination which may elucidate things a little here. As I remember it, it spoke of the word "imagination" as used in various parts of the Old Testament and New Testament, and although the meaning is not quite what we call imagination, it refers rather to a kind of thinking which is emotionalized in such a way that it would include such things as we commonly call rationalization. To use a modern psychological term, an intellectual type of thinking but one which is so emotionally distorted that it is rather untrustworthy, and I think that in that sense it fits in very nicely with what Dr. Barnes just said with regard to imagination.

THE MEANING OF MATHEMATICS

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A word of explanation may be in order concerning the inclusion of a paper with the above title on a program of the American Scientific Affiliation. Most of the papers presented at our previous conventions have been apologetic in character and contained only a relatively small amount of scientific information. As individual members of this affiliation we cannot be equally competent in the various areas of science including mathematics, astronomy, physics, chemistry, biology, sociology, geology, archaeology, anthropology, psychology and philosophy. It seems to me that we need a series of papers presented at our annual meetings, each of which will attempt to explain the essential character of that field of science so that those whose special training does not lie in that area might be aided in their understanding of the problems involved.

Since mathematics has been called "the queen of the sciences" and also their handmaiden and since its essential character is so very poorly understood by many, it seems desirable to delve into the mysteries of this subject. In the first place I would like to point out that mathematics is much more than the art of computation. This reminds me of the young mathematics student who had recently returned from Germany after spending three years there earning the doctor's degree. He was greeted by an old acquaintance in his home town who asked him what he had been doing while abroad. "Studying mathematics" was the reply. "Studying nothing but mathematics for three years" exclaimed his friend as he gazed at a brick wall across the street, "why I suppose you could count the bricks in that wall at a glance." This illustrates a popular misconception in regard to the nature of mathematics. To be able to compute with ease and to perform a few simple mathematical tricks is often taken to mean that one is gifted in mathematics. Such, however, is not necessarily true. In fact, some noted mathematicians have been rather slow at numerical computation.

A fair competence in manipulation is admitted to be a necessary prerequisite to understanding a mathematical argument. But no amount of technical facility will of itself teach anyone what mathematics is or what proof means; nor will it suggest what is probably the most important reason why mathematics is today an even more vital human need and social necessity than it was in the past. Manipulative skill may suffice for the average technician in the trades but it is inadequate as an aid to self-respecting citizenship in even a moderately intelligent society.

Now just what constitutes the essential character of mathematics? I think that we may say that the essential element is reasoning and that it is mainly deductive in character, although not exclusively so, since in the formulation of many theorems, we use the inductive approach.

First of all we start with elements, the undefined terms concerning which all our mathematical reasoning is done. *These elements have nothing whatsoever to do with the constituents of matter which are studied quite extensively in chemistry and physics.* In fact, as far as the mathematician is concerned, they have no necessary relation to anything in the world of the senses. They constitute the building blocks, quite few in number, out of which is built the entire methematical structure. Examples are number, point, line, and plane. These elements or objects or concepts are so fundamental in character as to be incapable of definition. In mathematics we freely admit that there are some things which we cannot define. These are the elements. This lack of definition for the elements is a most important point of distinction between mathematics and other fields of knowledge. Most, if not all, definitions found in the dictionary are circular ones. That is, they define an object in terms of the thing itself. Now, really, a circular definition is not a definition at all. For example the definition of the word number as found in Webster's new international unabridged dictionary: "The total aggregate or amount of unite (whether of things, persons or abstract units)". Here we find the word number defined in terms of aggregate or amount of units. But what meaning have these terms if not in terms of number?

After we have the elements of the subject, we next have definitions, axioms or postulates, propositions and theorems. The axioms are pure assumptions concerning the undefined elements. They may have been suggested by experience or they may have been chosen on the mere whim of some mathematician interested in seeing what he could make. In no sense are the postulates or axioms eternal truths or necessary; nor are they guaranteed by any extra human necessity or supernatural existence. The laying down of postulates is a free act of human beings.

The totality of the axioms of any branch of mathematics provides the implicit definition of all undefined terms in that area. For applications it is important that the concepts or elements and the axioms or postulates of mathematics correspond well with physically verifable statements about real tangible objects. The physical reality behind the concept of point is that of a very small object such as a pencil dot, while a straight line is an abstraction from a taut thread or of a ray of light. The properties of these physical points and straight lines are found by experience to agree more or less with the formal axioms of geometry. Quite conceivably more precise experiments might necessitate modification of these axioms if they are adequately to describe physical phenomena. But if the formal axioms did not agree more or less with the properties of physical objects, then geometry would be of little interest. Thus there is an authority, other than the human mind, that decides the direction of mathematical thought. *Correspondence with reality a matter of interest and usefulness, not of necessity

We usually require that the postulates be simple and not too great in number. Moreover, the postulates must be consistent, in the sense that no two theorems deducible from them can be mutually contradictory, and they must be complete, so that every theorem of the system is deducible from them. For reasons of economy it is also desirable that the postulates be independent, in the sense that no one of them is deducible from the others. The question of the completeness and of the consistency of a set of axioms has been the subject of much controversy. Different philosophical convictions concerning the ultimate roots of human knowledge have led to apparently irreconcilable views on the foundations of mathematics. If, as in the Kantian philosophy, mathematical entities are considered to exist in a realm of pure intuition, independent of definitions and of individual acts of the human mind, then of course there can be no contradictions, since mathematical facts are objectively true statements describing relations considered as real in the realm of pure intuition. From this intuitionist point of view there is no problem of consistency. Unfortunately, it has turned out that the intuitionist attitude, if applied without compromise, would exclude a large and important part of mathematics and would hopelessly complicate the rest. Radical intuitionists deny a legitimate place in mathematics to the number continuum. They completely reject all non-constructive proofs, and admit only the denumerably infinite as a legitimate child of intuition.

Perhaps we should add a word concerning the concept of the denumerably infinite. This brings in the added concept of one-to-one correspondence. Most of us have made use of this latter concept from our earliest attempts at counting material objects. As children many of us started to count by the use of our fingers. In fact, our word digit comes from the Latin, meaning finger. When you say that you have five objects and hold up five fingers to show the number, you are making use of the principle of one-to-one correspondence, which principle is very important in the consideration of infinite classes. In using this principle it is necessary to have two sets or classes of objects and then to add that if for every object in the first set there corresponds but one object in the second and conversely that for every object in the second there corresponds but one object in the first, then the number of objects in the two sets is the same. Now when we speak of the denumerably infinite we mean that we have a set which can be put into one-to-one correspondence with the set of the natural numbers: 1,2,3,4,... This set of the natural numbers is infinite, which simply means that if you name any number N, as large as you please, then there are still numbers in the set.

Quite different is the view taken by the formalists. They do not attribute an intuitive reality to mathematical objects, nor do they claim that axioms express obvious truths concerning the realities of pure intuition, their concern is only with the formal logical procedure of reasoning on the basis of postulates. This attitude has a definite advantage over intuitionism, since it grants to mathematics all the freedom necessary for theory and applications. But it imposes on the formalist the necessity of proving that his axioms, now appearing as arbitrary creations of the human mind, cannot possibly lead to a contradiction. Great efforts have been made during the last twenty five years to find such consistency proofs, at least for the axioms of arithmetic and algebra and for the concept of the number continuum. The results are highly significant, but success is still far off. Indeed, recent results indicate that such efforts cannot be completely successful, in the sense that proofs for consistency and completeness are not possible within strictly closed systems of thought. Remarkably enough, all these arguments on the foundations of mathematics proceed by methods that in themselves are thoroughly constructive and directed by intuitive patterns.

Let us consider a case where mathematical reasoning of the purely formalistic type has led to a contradiction. This involves the use of the concept of set without any restrictions being put upon it. This paradox, first shown by Bertrand Russell is as follows: Most sets do not contain themselves as elements. For example, the set A of all integers contains as elements only integers; A being itself not an integer but a set of integers, does not contain itself as an element. Such a set we may call ordinary. There may possibly be sets which do contain themselves as elements; for example, the set S defined as follows: "S contains as elements all sets definable by an English phrase of less than twenty words" could be considered to contain itself as an element. Such sets we might call extraordinary sets. However most sets will be ordinary, and we may exclude the erratic behavior of extraordinary sets by confining our attention to the set of all ordinary sets. Call this set C. Each element of the set C is itself a set; in fact, an ordinary set. The question now arises, is C itself an ordinary set or an extraordinary set? It must be one or the other. If C is ordinary, it contains itself as an element, since C is defined as containing all ordinary sets. This being so, C must be extraordinary, since the extraordinary sets are those containing themselves as members. This is a contradiction. Hence C must be extraordinary. But then C contains as a member an extraordinary set (namely C itself), which contradicts the definition whereby C was to contain ordinary sets only. Thus in either case we are led to a contradiction. What shall we do with the set C? I leave it to you to decide the matter.

Let us suppose that we have agreed upon some set of postulates or axioms for the undefined elements. One postulate for our points and lines might be, "two points determine a line," another, "two lines determine a point." The latter, by the way, would not usually be admitted in High School geometry, for in that subject are the exceptions introduced by parallels which, by definition, are lines having no point in common. But if we introduce an ideal point at infinity, all a matter of words without any clutter of mysticism, the postulate becomes intelligible without any exceptions.

Thus far we have the undefined elements and postulates about them. To the postulates we now apply common logic, or the laws of thought, and see what the postulates imply. The three so-called laws of thought of Aristotle are: (1) A is A (the law of identity), (2) nothing is both A and not-A (the law of excluded middle), (3) everything is either A or not A (the law of contradiction). These postulates of reasoning were once thought to be superhuman necessities and not, as they are regarded today, mere assumptions which human beings have made and agreed to accept. So let us refer to Aristotle's classical laws as the postulates of deductive reasoning. Deduction proceeds by an application of these postulates to those of the system, it may be geometry or algebra, which may be under investigation.

It is possible to make different kinds of assertions about the undefined elements. The most important of them are the propositions. A proposition is a statement which is either true or false. A true proposition is sometimes called a theorem. If true, we try to prove propositions by deductive reasoning. If false, an attempted deductive proof will sometimes reveal the falsity by the indirect method. Proof consists in seeing what the postulates of the system imply. Thus if P and Q are propositions and if Q follows from P by the postulates of deductive reasoning; and if further it is known or assumed that P is true, then Q is true. In particular, if P is one of our postulates which we have assumed at the beginning to be a true proposition, Q is true. But if it is not known whether Q is true, we may tentatively assume that it is false. If from this assumption we can deduce that Q is also true, we have a conflict with the postulate of excluded middle. But we agreed to abide by the postulates of deductive reasoning. To avoid the conflict we say that Q is not false, which we tentatively assumed; namely, Q is true, which we wished to prove.

The whole game is exceedingly simple. There are but two rules. First state all the postulates and second see that no other postulate slips into a chain of deductive reasoning. In geometry, for example, it looks as if a straight line which cuts one side of a triangle at a point other than a vertex must also cut another side. This is the sort of assumption which Euclid or some of his modern imitators might easily make. If it cannot be deduced from the remaining postulates it should be put in plain view with them as another postulate.

From the foregoing sketch of the nature of a mathematical system emerges the distinguishing feature of any such system, which is paradoxically stated in Bertrand Russell's epigram, "Mathematics is the science in which we never know what we are talking about nor whether what we say is true." The postulates from which everything starts are assumed to be true; to ask whether they are really true is to ask a question which is wholly irrelevent to the mathematics of the situation. The deductions from the postulates have the same truth value as the postulates themselves.

Although Russell's remark may tend to overemphasize the view of the older British school that mathematics is identical with logic; a view which, outside of Great Britain, is now generally regarded as untenable; it does call attention to a distinction between pure mathematics and applied mathematics. To see this, consider the statement often seen in elementary texts that the a,b,c,.....x,y,z of algebra represent numbers. Rather it should be stated that the letters are mere undefined marks or elements about which certain postulates are made. The very point of elementary algebra is simply that it is abstract, that is, devoid of any meaning beyond the formal consequences of the postulates laid down for the marks. Some of the elemen tary algebra is true when interpreted in terms of rational numbers; some of it is false for these same numbers; for example, the statement (which might be taken as a postulate in a first course) that every equation has a root. But we miss the whole point of algebra if we insist on any particular interpretation. Algebra stands on its own feet as a hypothetico-deductive system. An interpretation of the abstract system is an application.

To illustrate what has been said about mathematical systems let us glance at an elegant set of seven postulates for common algebra, from E.V. Huntington (Transactions of the American Mathematical Society, vol. 4, 1903, pp. 31-37). The system defined by these postulates is usaully called a field, and is identical, abstractly, with common, rational algebra. The fundamental concept involved is that of a class in which two rules of combination (or operations), denoted by $\emptyset$ b and a $\odot$ b are uniquely known elements of the class. This is sometimes expressed as "the class is closed under operations $\emptyset$, $\odot$." Neither a $\emptyset$ b nor a $\odot$ b belong to the class unless so stated explicitely. These remarks are merely by way of introduction; the postulates follow.

Postulate A1. If a, b and b$\emptyset$ a belong to the class, then a $\emptyset$ b = b $\emptyset$ a.

Postulate A2. If a, b, c, a $\emptyset$ b, b $\emptyset$ c, and a $\emptyset$ (b $\emptyset$ c) belong to the class, then (a $\emptyset$ b) $\emptyset$ c = a $\emptyset$ (b $\emptyset$ c).

Postulate A3. For every two elements a and b (a = b or a $\neq$ b), there is an element x such as a $\emptyset$ x = b.

Postulate M1. If a, b and b $\odot$ a belong to the class, then a $\odot$ b - b $\odot$ a.

Postulate M2. If a, b, c, a $\odot$ b, b $\odot$ c and a $\odot$ (b $\odot$ c) belong to the class, then (a $\odot$ b) $\odot$ c = a $\odot$ (b $\odot$ c).

Postulate M3. For every two elements a, b (a = b or a $\neq$ b), provided a $\emptyset$ a = a and b $\emptyset$ b $\neq$ b, there is an element y such that a $\odot$ y = b.

Postulate [illegible] If a, b, c, b $\emptyset$ c, a $\odot$ b, a $\odot$ c and (a $\odot$ b) $\emptyset$ (a $\odot$ c) belong to the class, then a $\odot$ (b $\emptyset$ c) = (a $\odot$ b) $\emptyset$ (a $\odot$ c).

The unusual $\emptyset$, $\odot$ instead of the familiar +, x are used to prevent any possible misconception that we are talking about numbers as in arithmetic. We are not; the marks or undefined elements a, b, c ... are marks and nothing more, and the seven postulates state everything that we are assuming about these marks and $\emptyset$, $\odot$. It is easy to see, as already suggested, that the postulates define common school algebra (including the ban against attempting to divide by zero) up to the point where radicals are introduced. Perhaps the complete freedom, the arbitrariness of what we are doing will be more obvious when we realize that the seven postulates are independent of one another. That is, it is possible to exhibit a system which does not satisfy any particular one of of the sven postulates, but which does satisfy the remaining six. You may easily verify that the set of all positive rational numbers with a $\emptyset$ b and a $\odot$ b now defined to be b and ab respectively, that is, a $\emptyset$ b equals b and a $\odot$ b equals a, b, satisfies all the postulates except A1. In the same way, a system satisfying all except M1 is the system of all integral numbers with a $\emptyset$ b - a + b and a $\odot$ b = b. A system which satisfies all integral numbers with a $\emptyset$ b = a + b and a $\odot$ b = a + b. Perhaps you are getting bored with this discussion so I shall pass on to two very fine cases of mathematical reasoning which more nearly approach everyday experience.

The first case is that of the proof of the theorem which states that the number of prime numbers is infinite. As you will recall, a prime number is one divisible only by itself and unity. By the number of such numbers being infinite is meant that if one attempts to name any very large number as representing all the primes, that there are sti more primes to be found. The prooof is as follows. Let us assume that a largest prime number exists. C ll it P. Now let us form a number N as equal to the product of all the primes from the first one which is two to the last one which we have assumed to be P and then let us add one to this product. Therefore N = (a.3.5.7.............P) + 1. Now the question is whether this number is prime numbers starting with two and ending with P. Hence we see that this number.

is not divisible by any number other than itself and unity. Therefore it is prime and surely larger than P which we had assumed to be the largest prime. Hence we have a contradiction and we conclude that there does not exist a largest prime number and therefore the number of prime numbers is infinite.

The next case we wish to prove is that the square root of the number two is irrational, that is, it is a real number which is not the quotient of two integers. To prove this let us assume that the square root of two is rational and denote it by the ratio p/q, where p and q have no common divisor, that is, they have been reduced to lowest terms. Now square both sides of the equality and we have $2 = p^2/q^2$ or $p^2 = 2q^2$. Here we have applied the axiom that when equals are multiplied by equals the results are equal. If, however, $p^2 = 2q^2$, then p^2 is even and hence p is even since only even numbers when squared result in even numbers. If p is even we can express it in the form $p = 2n$. From this equation we have $p^2 = 4n^2$, again by the application of the axiom that when equals are multiplied by equals the results are equal. Now applying the axiom that things equal to the same thing are equal to each other we have $2q^2 = 4n^2$ or $q^2 = 2n^2$. This latter equation states that q^2 is even and hence q is even. This is a contradiction with out previous statement that p and q have no common factor since, if they are both even, they have the common factor two. Since our assumption that the square root of two was rational has led us to a contradiction, we conclude that the square root of two is irrational and the proof is complete. In both of these cases we have used the method of proof known as reductio-ad-absurdum.

We come now to the question of the application of mathematics to the physical world and to all of God's creation. We have emphasized the fact that pure mathematics is an invention of the human mind. The more nearly that the elements and the postulates of the subject correspond with what we consider as physical realities, the more closely will the results correspond and can be used to predict the manner in which physical phenomena will perform. Hence as was stated in the beginning of this paper, mathematics becomes the handmaiden or servant of the sciences. But just to say that something has been proved mathematically, does not insure that the results will correspond with physical phenomena. For example, it was proved a number of years ago that it was impossible for a heavier-than-air craft to fly through the air. This is no discredit to mathematics, but rather it serves to warn us that we need to be very careful in the applications of mathematics. Since all sciences are continually making more and more use of mathematical methods, we need to keep constantly on our guard that the results obtained correspond as closely as possible with physical phenomena.

Finally I would like to say a word concerning the relationship existing between mathematics and the Christian idea of God. Since I believe with Professor Jaarsma in his paper entitled, "Christian Theism and the Empirical Sciences" that "the God of Christianity as the Creator is the unconditioned Conditioner of all things, including the very facts and conclusions of science," I feel that even the thoughts of mathematicians have their ultimate source in God. However to say, as some have said, "that the Great Architect of the Universe now begins to appear as a pure mathematician," appears to me to belittle the idea of God. The pure mathematician is just a puny little man with a quite finite mind doing a small bit of purely human reasoning. If some of this reasoning does seem to aid us in delving into the mysteries of God's creation, we should give more glory to His name for allowing us this privilege. But to put the infinite God, creator and sustainer of the universe, as well as savior of our souls, into this category seems to me to be quite a serious blunder. May we then, as Christian men of sicnece, make more use of the mathematical method in science, since it has proved so fruitful in leading us into a deeper understanding of God's creation.

DISCUSSION OF DR. HARTZLER'S PAPER

Dean Miller: Thank you again, Dr. Hartzler. Now who has the questions written down on the back of the envelop for this one? I dare you to ask him.

Dr. Bender: Maybe these postulates are purely arbitrarily chosen, but I think the fact reamins that Euclid's postulates are not arbitrarily chosen.

Dr. Hartzler: I don't know how he chose them, yet I think they are chosen in such a way to agree with the space relationships that we know of, so that the results of the Euclidian geometry and the reasons for the Euclidian geometry are usable, and the engineer can use them. Let's put it this way, is it correct to say that the postulates of the particular mathematical systems that we ordinarily use in applied mathematics, such as geometry and the number system---these postulates for these systems have been arrived at inductively by determining what postulates would be needed in the system which we are familiar with?

For instance, in the number system, we learn that A plus B equals B plus A by inductive reasoning. We try it by using several sets of numbers and finally by inductive reasoning we arrive at a generalization, at that conclusion regarding A plus B we may use it as a postulate in that system; and when we have our postulates chosen in that way, then we have the application that we want to use in applied mathematics.