The Final Merger of Comparable Mass Binary Black Holes

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Abstract. A remarkable series of breakthroughs in numerical relativity modeling of black hole binary mergers has occurred over the past few years. This paper provides a general overview of these exciting developments, focusing on recent progress in merger simulations and calculations of the resulting gravitational waveforms.

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INTRODUCTION

The coalescence of a comparable mass black hole binary (BHB) is a powerful source of gravitational waves and proceeds in three phases: inspiral, merger, and ringdown [1]. The inspiral stage, when the BHs are widely separated and following quasi-circular trajectories, and the ringdown stage, during which the final merged BH settles into a quiescent Kerr state, can both be treated analytically. However the merger phase, in which the two BHs plunge together and merge to form a highly distorted remnant BH, occurs in the arena of very strong, dynamical gravitational fields and can only be calculated using numerical relativity.

This final merger will produce an intense burst of gravitational radiation with a luminosity $\sim 10^{23} L_{\odot}$, briefly emitting more energy than the combined light from all the stars in the visible universe. Such bursts are expected to be among the strongest sources for LISA, which will observe mergers of massive BHBs. Mergers of stellar mass and intermediate mass BHBs are likely to be the strongest sources for ground-based gravitational wave detectors such as LIGO and VIRGO. Observing the gravitational waves from the final merger will allow unprecedented tests of general relativity in the dynamical, strong field regime – provided we know the waveforms that general relativity predicts.

The final merger of BHBs also has compelling astrophysical implications. In particular, when the BHs have unequal masses, the resulting gravitational wave emission is asymmetric; since the gravitational waves carry momentum, the merged remnant BH suffers a recoil kick [2]. If this kick is large enough, it could eject the merged remnant from its host structure, thereby affecting the overall rate of merger events [3]. In addition, since BHBs are generally expected to be spinning, their mergers could produce interesting spin dynamics and couplings [4].

For more than 30 years, numerical relativists have attempted to calculate the merger of comparable mass BHs and the resulting gravitational waveforms. This has proved to be a very difficult undertaking indeed. In particular, the simulation codes have been plagued by a host of difficulties, typically resulting in various instabilities that caused them to crash before any sizeable fraction of a binary orbit could be evolved.

Recently, however, a series of dramatic breakthroughs has occurred in numerical relativity, resulting in accurate and robust simulations of BHB mergers and their gravitational waveforms. This paper provides a general overview of these exciting developments. We begin with a brief overview of numerical relativity and BHB calculations. The heart of the paper then focuses on recent progress in computing BHB orbits and mergers, and the resulting waveforms. The paper concludes with a summary and outlook for the future.

With the goal of reaching as wide an audience as possible, technical details are deliberately kept to a minimum; interested readers can find more detailed information in the references.[1] We follow conventional practice by setting $G = 1$ and $c = 1$, which allows us to measure both time and distance in terms of mass M. In particular, $1M \sim (5\times 10^{-6})(M/M_\odot)\mathrm{sec} \sim 1.5(M/M_\odot)\mathrm{km}$. Spatial indices are taken to have the range $i = 1,2,3$. Note that the simulation results scale with the masses of the BHs, and are thus applicable to LISA as well as to ground-based detectors.

NUMERICAL RELATIVITY

Numerical relativists construct a spacetime by solving the Einstein equations on a computer. In the most commonly used "3+1" [5, 6] approach, 4-D spacetime is considered to be sliced into a stack of 3-D spacelike hypersurfaces labeled by time t. The main independent variables are essentially the 3-metric g_{ij} and its first time derivative $\partial_t g_{ij}$ on each slice. The equations split naturally into two sets. The constraint equations provide relationships that must be satisfied at any time t; in particular, initial data for BHBs is set by solving the constraint equations on a 3-D slice at some initial time $t = 0$. The evolution equations are used to propagate this data forward in time. The four coordinate degrees of freedom in general relativity give four freely-specifiable coordinate or gauge conditions for the future development of the time and spatial coordinates. During the evolution the gauge is specified by the lapse function α, which gives the lapse of proper time $\alpha\Delta t$ between neighboring slices, and the shift vector β^i, which governs how the spatial coordinates develop from one slice to the next.

Efforts to evolve the merger of two BHs have a long history.[2] The first attempt to solve the Einstein equations on a computer was carried out by Hahn and Lindquist in 1964 [7], who tried to evolve the head-on collision of two equal mass BHs. (Since the term "black hole" had not yet been coined, they called their paper "The two-body problem in geometrodynamics.") Due in part to a poor choice of coordinate conditions, the

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[1] Since this paper is not a full review of the subject, the reference list is representative rather than comprehensive. We attempted to cite key papers from the major numerical relativity efforts that were steppingstones to the current breakthroughs.

[2] An excellent history of the developments in numerical relativity treatments of the BHB problem can be found in the talk by Miguel Alcubierre at the Astrophysical Applications of Numerical Relativity workshop held in May 2006 in Guanajuato, Mexico.

evolution crashed shortly after it began. In the mid-1970s, Smarr and Eppley [8, 9, 10] pioneered the use of the the 3+1 approach with improved coordinate conditions, including conditions on the lapse function to produce slices that avoid crashing into singularities [11, 12]. Although their simulations encountered instabilities and had problems with accuracy, they were able to evolve the head-on collision and extract some information about the resulting gravitational waves. Following this significant achievement there was very little work on BHB simulations throughout the 1980s, although some numerical relativity work did continue, mostly on neutron stars.

In the 1990s work on ground-based gravitational wave detectors such as LIGO moved ahead strongly; since BHB mergers are considered one of the most promising sources for these detectors, numerical relativity work on the BHB problem started up again. More accurate calculations of head-on collisions were carried out, starting in axisymmetry [13]. In the mid-1990s, the NSF funded the Binary Black Hole Grand Challenge collaboration, a large multi-institution effort aimed at evolving BHBs in 3-D and calculating the resulting gravitational wave signatures. A vigorous numerical relativity program was also started at the newly-formed Albert Einstein Insitutue in Germany. While many important developments resulted from this era, including the development of large 3-D codes and the ability to evolve boosted BHs [14] and grazing collisions [15, 16, 17], the problem turned out to be more difficult than anticipated and the codes were plagued by instabilities that caused them to crash.

During the late 1990s and early 2000s, the ground-based detectors began taking data, the importance of BHB mergers as sources for LISA grew, and new research groups arose in numerical relativity. The role of unstable modes present in the formulations of the numerical relativity equations was recognized as a major issue. Work on key areas such as gauge conditions, formalisms, boundary conditions, and the role of the constraints in evolutions was carried out. Overall, progress in obtaining stable 3-D BH evolutions was slow and incremental. The Lazarus approach combined a brief 3-D numerical relativity evolution of a BHB near the final plunge with a late-time perturbative evolution to make use of the short-duration stable evolutions that were then possible, and produced the first gravitational waveform from a BHB [18, 19].

Most of the recent work in numerical relativity has been carried out using a conformal formulation of the Einstein equations known as BSSN [20, 21]. In this approach, the set of evolution equations has first-order time derivatives and second-order spatial derivatives, and is strongly hyperbolic [22, 23]. The constraint equations have been incorporated into the evolution equations to improve the performance. In some cases the BHs are represented as "punctures," with the singular parts being factored out [24]; in other cases, the BH interiors are excised to remove the singular parts from the grid [25, 26]. Various gauge conditions were developed to allow longer evolutions, including new slicing conditions that avoid evolving into a singularity and shift conditions that prevent the coordinates from falling into the BHs [27]. Other approaches feature fully first-order symmetric hyperbolic formulations of the Einstein equations and special attention to constraint preserving boundary conditions [28].

For successful BHB merger simulations, it is necessary to resolve both the BHs (with spatial scales $\sim M$) and extract the gravitational radiation (with scales $\lambda_{\rm GW} \sim (10-100)M$) in the wave zone. Since the large 3-D codes strain the capacities of current high performance computing facilities, this requires the use of variable resolution within the computational domain. Most of the current numerical relativity codes use finite differences on a 3-D Cartesian grid with fixed or adaptive mesh refinement. There are also a few efforts that use spectral methods, which also incorporate variable resolution.

In the past two years, there has been significant and rapid progress in numerical relativity simulations of BHB mergers across a broad front. The first complete orbit of a BHB was achieved in 2004. This was followed shortly by the first simulation of a BHB through an orbit, plunge, merger and ringdown. Since late 2005, new ideas have opened the field up even more, and the past year has seen dramatic progress. These developments are reviewed in the next section.

BHB ORBITS AND MERGERS

The first complete orbit of an equal mass, nonspinning BHB binary was achieved by Brügmann, Tichy, and Jansen [29] using the standard conformal BSSN approach with the BHs represented as "punctures" [24]. The 3-metric on the initial slice is written as $g_{ij} = \psi^4 \delta_{ij}$, where the conformal factor $\psi = \psi_{\rm BL} + u$ and $i, j = 1, 2, 3$. The static, singlar part of the conformal factor has the form $\psi_{\rm BL} = 1 + \sum_{n=1}^{2} m_n / 2|\vec{r} - \vec{r}_n|$, where the $n^{\rm th}$ puncture black hole has mass m_n and is located at $\vec{r}_n$. The nonsingular function u is obtained by solving one of the constraint equations.

In the standard puncture approach, the singular part of the metic is factored out and handled analytically during the evolution, and only the regular parts are evolved numerically. This requires that the punctures remain fixed on the numerical grid, resulting in a stretching of the coordinate system and the development of large errors in the metric as the binary evolves. Excision of the regions around the punctures (but within the horizons) can be used to reduce errors and prolong these runs; excision can be applied to the individual punctures as well as inside a common horizon at late times.

For head-on and grazing collisions starting from relatively close separations, these methods work fairly well, as a common horizon forms quickly and excision prevents the unbounded growth of errors and allows the simulations to continue long enough for the BHs to merge. For orbiting BHs, a corotating coordinate frame implemented by an angular shift vector is needed since the punctures remain fixed on the grid. This can cause serious problems, however, such as superluminal coordinate speeds at large distances from the BHs and incoming noise from the outer boundary of Cartesian grids.

Brügmann, Tichy, and Jansen [29] introduced comoving coordinates using a shift vector that is dynamically adjusted during the evolution of the BHB to minimize both the angular and radial motion of the BHs. Their code uses fixed mesh refinement implemented by nested Cartesian boxes, with the resolution decreasing for boxes that span successively larger regions of the domain. They were able to evolve a BHB using excised punctures for more than one orbit, to $t \sim 185M$, where M is the total mass of the system. The code did crash before the BHs merged and inaccuracies in the outer regions prevented the extraction of gravitational waves. Their paper first appeared as a preprint in December 2003. Later, more accurate work by Diener, et al.[30] highlights the importance of high resolution and the effects of gauge choices on the resulting evolutions. Nevertheless, this first simulation of a full BHB orbit was a major step forward.

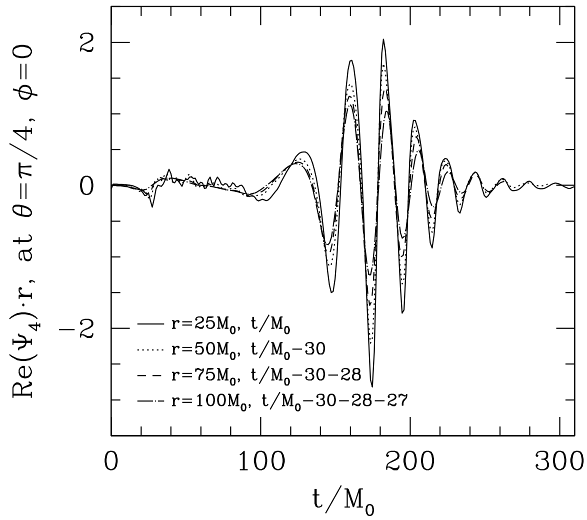

FIGURE 1. Waveforms from Pretorius' simulation [31].

In the first part of 2005, Pretorius carried out the first evolution of a BHB through a single plunge orbit, merger and ringdown [31] using an approach completely different from the standard one [32, 33]. Instead of using the 3+1 technique, he evolves the 4-metric directly, using generalized harmonic coordinates. The evolution equations have second-order time derivatives and constraint damping terms designed to remove spurious non-physical solutions. Numerical dissipation is added to control high frequency instabilities.

His initial data consists of two Lorentz-boosted scalar field profiles, with positions and velocities chosen to approximate a BHB orbit. Each scalar field configuration quickly collapses to form a BH, yielding a BHB system. The BH interiors are excised and the BHs move freely across the grid as the binary evolves; no corotating coordinates are needed. Adaptive mesh refinement is used to provide higher resolution in the regions near the BHs. Outside the orbital region, fixed mesh refinement is implemented using nested Cartesian boxes centered on the origin.

In Pretorius' simulation the BHs plunge together, completing ~ 1 orbit before merging. The code continues to run stably during the subsequent ringdown of the merged remnant BH, allowing the emitted gravitational waves to propagate outward far enough to be extracted. Figure 1 is taken from Ref. [31] and shows the gravitational waveforms from this simulation as represented by $r\Psi_4$, where Ψ_4 is the real part of the Weyl tensor component and r is the coordinate distance from the center of the grid.[3]

This achievement set a new standard in numerical relativity. And the use of these nonstandard techniques raised the important question of whether such methods were essential for successful BHB merger simulations.

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[3] For comparison with the waveforms shown in Figure 3, note that Pretorius uses the time coordinate t/M_0, where M_0 is the mass of a single BH. Also, the amplitude of $r\Psi_4$ has not been scaled with M; when this is done, the amplitude is comparable with that shown in Figure 3.

In late 2005 this question was answered when the numerical relativity groups from the University of Texas at Brownsville (UTB) [34] and NASA's Goddard Space Flight Center [35] simultaneously and independently discovered a way to evolve a BHB through an orbit, plunge, and merger within the 3 + 1 approach. Both of their codes are based on the BSSN formalism, with a key difference. Initially the BHs are set up using the standard puncture technique. However, during the evolution the singular part of the conformal factor is *not* factored out but rather is evolved together with the nonsingular part, using regularization to handle the puncture singularities. This allows the puncture BHs to move freely across the grid; no excision or corotating coordinates are needed.

Important ingredients in the success of the moving puncture method are the novel coordinate conditions for the lapse and shift. The UTB and Goddard groups each developed somewhat different lapse and shift conditions; both codes produced stable and accurate evolutions of a BHB through the final plunge, merger, and ringdown. The UTB code used a particular coodinate transformation to allow the grid resolution to vary smoothly over the computational domain, whereas the Goddard code employed fixed mesh refinement based on nested Cartesian boxes. Both groups were able to extract accurate and convergent gravitational waveforms.

The first results from moving puncture evolutions were presented by the UTB and Goddard groups in early November 2005 at the "Numerical Relativity 2005" workshop,[4] and the papers were submitted shortly thereafter. Since these techniques were developed within the widely-used traditional 3+1 numerical relativity approach, they could be readily adopted by other groups with similar 3-D BSSN codes. Indeed, in early January 2006 the Penn State group submitted a paper applying these techniques to evolutions of nonequal mass BHBs [36]. The UTB and Goddard groups moved quickly and were soon able to evolve multiple orbits followed by merger and ringdown [37, 38, 39]. At the April 2006 APS meeting, an entire session was devoted to BHB merger simulations with moving punctures. More groups adopted the moving puncture technique and, by the summer of 2006, this method was being actively used, studied, and advanced by the majority of the numerical relativity community working on BHB simulations.

In the early spring of 2006, the Goddard group used the moving puncture method to study the dynamics and radiation generation in the last few orbits and merger of an equal mass nonspinning BHB [38]. Using adaptive mesh refinement to follow the BHs in the orbital region and fixed mesh refinement based on nested Cartesian boxes in the outer regions, they ran a series of long duration runs starting from successively wider separations. In each case, the BHs start out on approximately circular orbits; key parameters for these runs are given in Table 1. Here, L/M_0 is the initial proper separation of the BHs,[5] T_{sim} is the duration of each run, and T_{merger} is the time at which the merger occurs, starting from the initial time in each run. The number of orbits N_{orbits} is estimated from the trajectories; see Figure 2.

Note that these initial data sets are an approximation to the actual conditions of a BHB in the real universe. Astrophysically, a BHB quickly circularizes and then spirals together, emitting gravitational waves, through $> 10^3$ nearly circular orbits before the

[4] The presentations from this meeting are posted at http://astrogravs.gsfc.nasa.gov/conf/numrel2005/.
[5] For the Goddard runs, M_0 is the initial total system mass.

TABLE 1. Parameters of long duration BHB runs calculated by the Goddard group using the moving puncture method [38].

Run	L/M_0	$T_{\rm sim}$	$T_{\rm merger}$	$N_{\rm orbits}$
R1	9.9	$421M$	$160M$	1.8
R2	11.1	$531M$	$234M$	2.5
R3	12.1	$530M$	$396M$	3.6

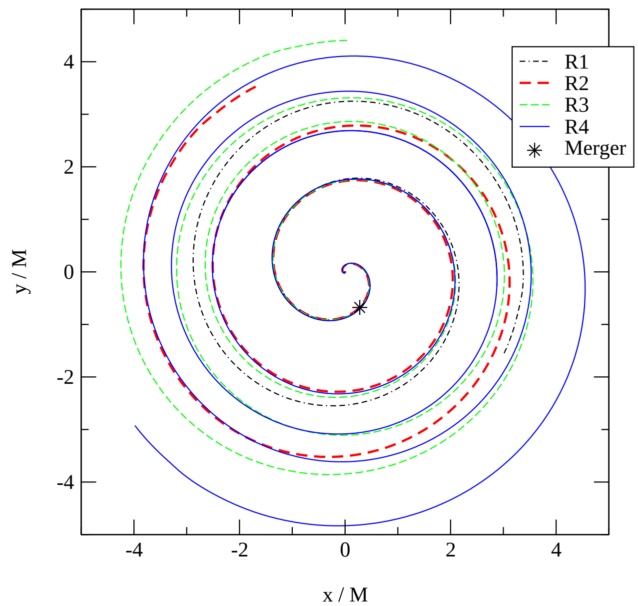

FIGURE 2. Paths of BHs for the long duration runs by the Goddard group, with parameters given in Table 1. For clarity, only the track of one BH from each run is shown [38].

final plunge and merger. Ideally, one would start a numerical relativity simulation of a BHB with initial data that uses BH positions and velocities from an inspiralling astrophysical orbit, and includes the outgoing gravitational radiation from earlier parts of the inspiral. If this were possible, one could carry out a sequence of simulations, starting the BHs with successively wider separations, and expect the BHs to follow the same astrophysical trajectories all the way through the merger. Unfortunately, there are currently no methods available for setting up such astrophysically accurate initial data. All existing methods produce BHB initial data with various spurious effects that deviate somewhat from the desired astrophysical data. However, if these deviations are small enough, they should disappear as the BHs spiral together, leading to the correct astrophysical trajectory predicted by the Einstein equations.

The sequence of runs by the Goddard group clearly demonstrates this behavior. Figure 2, which is taken from Ref. [38], shows the trajectories followed by the punctures in their four runs; for clarity, only the track of one of the BHs from each simulation is shown. Run R4 has the widest initial separation and completes the most orbits. After an

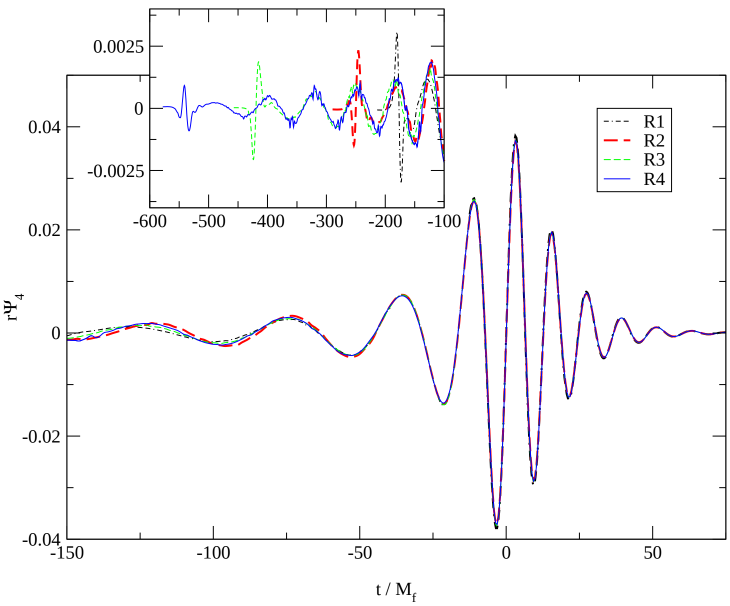

FIGURE 3. Waveforms from the long duration runs by the Goddard group, with parameters given in Table 1 [38]

initial transient period of approximately one orbit, the trajectory from each of the other runs nearly locks on to the R4 trajectory. For the final orbit, the trajectories from all of the runs are very nearly superposed. The fact that the BH paths lock on to a common universal trajectory for the final orbit and thereafter supports the idea that the late-time dynamics is dominated by the strong-field interactions and radiative losses; this has the effect of reducing the dependence on the initial conditions.

This universal dynamics produces a universal gravitational waveform. Figure 3, which is taken from Ref. [38], shows one polarization component of $r\Psi_4$, where Ψ_4 is the Weyl tensor component. The waveforms have been normalized so that the peak amplitude of the gravitational radiation occurs at $t = 0$. These waveforms all agree to within 1% for the last orbit, merger and ringdown (after $t \sim -50M_f$) and, except for a brief initial pulse at the beginning of each run, to within $\sim 10\%$ for the preceeding few orbits (shown in the inset).

The Goddard simulations consistently produce a final merged remnant BH with spin $a/m = 0.69$ to within $\sim 1\%$. The amount of energy released as gravitational waves varies slightly, since the simulations have different durations. For the longest run R4, they find $E_{\rm rad} = 0.039M$.

The discussion so far has focused on mergers of nonspinning BHs. Astrophysical BHs, however, are expected to be spinning. The UTB group was the first to carry out merger simulations of equal mass BHBs with spins [40]. Using the moving puncture method, they evolved BHs with equal spins, $a = 0.75m$, where m is the individual BH mass. They considered two cases: both BH spins aligned with the orbital angular momentum, and both spins anti-aligned. In both cases, the BHs started out on quasi-circular orbits with period $125M$.

The BHB in the anti-aligned case undergoes a prompt merger, completing ~ 0.9 orbits

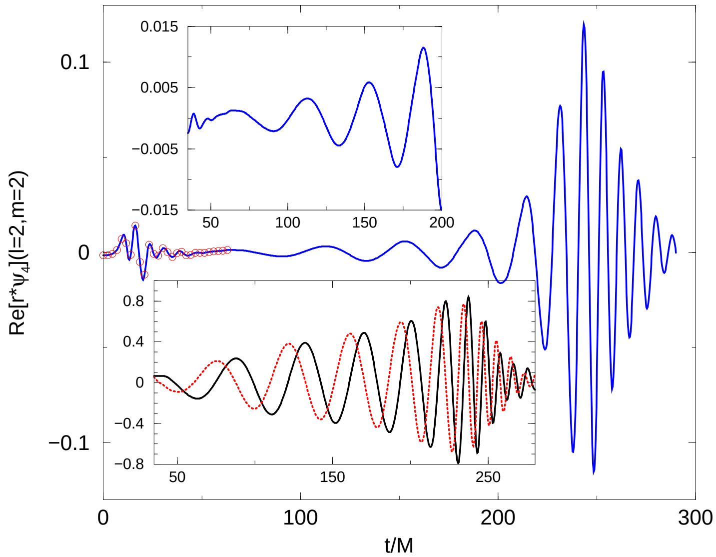

FIGURE 4. Waveforms from the UTB simulation of merging BHs with spins aligned with the orbital angular momentum [40]. The top inset shows the early part of the waveform. The bottom inset shows the real (solid) and imaginary (dotted) components of the $l = 2, m = 2$ component of the strain h calculated at $r = 10M$.

before a common horizon forms, to yield a remnant BH with spin $a \sim 0.44M$. In the aligned case, the initial total angular momentum (orbital + spin) $> M$, and the binary completes ~ 2.8 orbits before merging to form a BH with $a \sim 0.9M$. In this case, it appears that the merger temporarily stalls as the excess angular momentum is radiated away, in order to form a final Kerr BH with $a < M$. Figure 4, taken from Ref. [40], shows the gravitational waveform from this aligned case. In both the aligned and anti-aligned cases, the resulting gravitational waveforms show a simple shape, similar to that seen in the non-spinning case; *c.f.* Figure 3

Finally, astrophysical BHBs are also expected to have unequal masses, especially in the case of the massive BHBs that LISA will observe. As mentioned above, the gravitational wave emission will be asymmetric in this case, imparting a recoil kick to the merged remnant BH. Although post-Newtonian techniques have been used to calculate the kick velocity during the inspiral [41, 42], almost all of the recoil comes from the strong gravity regime. Numerical relativity simulations are therefore needed for an accurate calculation of the kick velocity. Unequal mass merger calculations are technically more demanding than the equal mass case, due in part to the need to resolve the smaller BH which moves faster than the larger one. In addition, getting the correct value for the kick requires a sensitive calculation arising from higher-order gravitational wave modes.

The Penn State group was the first to calculate mergers of unequal mass nonspinning BHBs, employing the moving puncture method [36]. Using fairly low resolution and starting their BHs at relatively close separations, they ran several different mass ratios in the range $1 \leq m_1/m_2 \leq 0.32$, and quote lower limits on the kick velocities. More recently, the Goddard group carried out more accurate calculations with higher resolution for the case $m_1/m_2 = 0.67$ [43]. They also examined the dependence of the resulting kick velocity on the initial separation of the BHs. Using higher resolution and adaptive mesh refinement, they estimate the astrophysically relevant range of kick values to be $(86-97)\mathrm{km/s}$ for this mass ratio.

SUMMARY AND FUTURE OUTLOOK

The past few years have seen a remarkable series of breakthroughs in numerical relativity modeling of BHB mergers. The first BHB orbit was achieved by Brügmann, Tichy, and Jansen [29] using special comoving gauge conditions in a traditional numerical relativity code in late 2003. Roughly a year and a half later, the first plunge, merger, and ringdown calculation by Pretorius appeared, using nonstandard techniques and including the extraction of the gravitational waveform [31].

Less than six months later, in November 2005, the moving puncture method was introduced by the UTB [34] and Goddard [35] groups, enabling merger calculations with simple but novel gauge conditions in traditional numerical relativity codes. This was rapidly exploited by its developers and, by the early spring of 2006, the Goddard group had obtained a consistent solution for the gravitational wave burst from the merger of two equal mass Schwarzschild BHs, independent of the (quasi-astrophysical) initial conditions [38]. Very shortly thereafter, the UTB group produced the first evolutions of equal mass, spinning BHB mergers, demonstrating the orbital hangup when the BH spins are aligned with the orbital angular momentum [40].

The first attempt to model unequal mass mergers using the moving puncture method was carried out by the Penn State group [36] and first appeared in January 2006. This was soon followed by similar work at higher resolution and larger initial separations by the Goddard group [43]. Overall, by the summer of 2006, a large segment of the numerical relativity community was actively adopting, adapting, and exploiting the moving puncture method.

At the present time there is broad consensus that the merger of two equal mass Schwarzschild BHs produces a final remnant BH with spin $a \sim 0.7M$, and that the amount of energy radiated in the form of gravitatational waves, starting with the final few orbits and proceeding through the plunge, merger and ringdown, is $\sim 0.04M$. The UTB and Goddard groups are working with Pretorius to compare their waveforms; preliminary results using longer runs by the UTB group and Pretorius show good agreement with the waveforms obtained by the Goddard group shown in Figure 3. Plans are underway to include other groups in this comparison effort.

The outlook for continued progress in BHB merger simulations is very bright. New work with moving punctures continues to appear [44, 45, 46]. Pretorius is carrying out new and longer runs with his generalized harmonic code. The Caltech-Cornell collaboration has made important progress in carrying out orbits with their spectral code based on a fully first-order formulation of the Einstein equations [47] and hopes to achieve mergers soon. The impressive progress in this field, occurring across a broad front, is very encouraging. Stay tuned!

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REFERENCES

1. Éanna É. Flanagan, and S. A. Hughes, *Phys. Rev. D* **57**, 4535 (1998).
2. M. Favata, S. A. Hughes, and D. E. Holz, *Astrophys. J.* **607**, L5–L8 (2004), `astro-ph/0402056`.
3. D. Merritt, M. Milosavljevic, M. Favata, S. A. Hughes, and D. E. Holz, *Astrophys. J.* **607**, L9–L12 (2004), `astro-ph/0402057`.
4. D. Merritt, and R. D. Ekers, *Science* **297**, 1310–1313 (2002), `astro-ph/0208001`.
5. R. Arnowitt, S. Deser, and C. W. Misner, "The Dynamics of General Relativity," in *Gravitation: An Introduction to Current Research*, edited by L. Witten, John Wiley, New York, 1962, pp. 227–265.
6. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco, 1973.
7. S. G. Hahn, and R. W. Lindquist, *Ann. Phys.* **29**, 304–331 (1964).
8. L. Smarr, A. Čadež, B. DeWitt, and K. Eppley, *Phys. Rev. D* **14**, 2443–2452 (1976).
9. L. Smarr, *Ann. N. Y. Acad. Sci.* **302**, 569–604 (1977).
10. L. Smarr, "Gauge Conditions, Radiation Formulae and the Two Black Hole Collision," in *Sources of Gravitational Radiation*, edited by L. Smarr, Cambridge University Press, Cambridge, England, 1979, p. 245.
11. L. Smarr, and J. York, *Phys. Rev. D* **17**, 2529 (1978).
12. L. Smarr, and J. York, *Phys. Rev.D* **17**, 1945 (1978).
13. P. Anninos, D. Hobill, E. Seidel, L. Smarr, and W.-M. Suen, *Phys. Rev. D* **52**, 2044–2058 (1995).
14. G. B. Cook, et al., *Phys. Rev. Lett* **80**, 2512–2516 (1998).
15. B. Brügmann, *Int. J. Mod. Phys. D* **8**, 85 (1999), `gr-qc/9708035`.
16. S. Brandt, R. Correll, R. Gómez, M. F. Huq, P. Laguna, L. Lehner, P. Marronetti, R. A. Matzner, D. Neilsen, J. Pullin, E. Schnetter, D. Shoemaker, and J. Winicour, *Phys. Rev. Lett.* **85**, 5496–5499 (2000).
17. M. Alcubierre, W. Benger, B. Brügmann, G. Lanfermann, L. Nerger, E. Seidel, and R. Takahashi, *Phys. Rev. Lett.* **87**, 271103 (2001), `gr-qc/0012079`.
18. J. Baker, B. Brügmann, M. Campanelli, C. O. Lousto, and R. Takahashi, *Phys. Rev. Lett.* **87**, 121103 (2001), `gr-qc/0102037`.
19. J. Baker, M. Campanelli, C. O. Lousto, and R. Takahashi, *Phys. Rev.* **D65**, 124012 (2002), `astro-ph/0202469`.
20. M. Shibata, and T. Nakamura, *Phys. Rev. D* **52**, 5428 (1995).
21. T. W. Baumgarte, and S. L. Shapiro, *Phys. Rev. D* **59**, 024007 (1999), `gr-qc/9810065`.
22. G. Nagy, O. E. Ortiz, and O. A. Reula, *Phys. Rev.* **D70**, 044012 (2004), `gr-qc/0402123`.
23. O. A. Reula (2004), `gr-qc/0403007`.
24. S. Brandt, and B. Brügmann, *Phys. Rev. Lett.* **78**, 3606–3609 (1997), `gr-qc/9703066`.
25. D. Shoemaker, et al., *Class. Quant. Grav.* **20**, 3729–3744 (2003), `gr-qc/0301111`.
26. M. Alcubierre, et al. (2004), `gr-qc/0411137`.
27. M. Alcubierre, B. Brügmann, P. Diener, M. Koppitz, D. Pollney, E. Seidel, and R. Takahashi, *Phys. Rev. D* **67**, 084023 (2003), `gr-qc/0206072`.
28. L. Lindblom, et al., *Phys. Rev.* **D69**, 124025 (2004), `gr-qc/0402027`.
29. B. Bruegmann, W. Tichy, and N. Jansen, *Phys. Rev. Lett.* **92**, 211101 (2004), `gr-qc/0312112`.
30. P. Diener, et al., *Phys. Rev. Lett.* **96**, 121101 (2006), `gr-qc/0512108`.
31. F. Pretorius, *Phys. Rev. Lett.* **95**, 121101 (2005), `gr-qc/0507014`.
32. F. Pretorius (2006), `gr-qc/0602115`.
33. F. Pretorius, *Class. Quant. Grav.* **22**, 425–452 (2005), `gr-qc/0407110`.

34. M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower, *Phys. Rev. Lett.* **96**, 111101 (2006), `gr-qc/0511048`.
35. J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter, *Phys. Rev. Lett.* **96**, 111102 (2006), `gr-qc/0511103`.
36. F. Herrmann, D. Shoemaker, and P. Laguna (2006), `gr-qc/0601026`.
37. M. Campanelli, C. O. Lousto, and Y. Zlochower, *Phys. Rev. D* **73**, 061501 (2006), `gr-qc/0601091`.
38. J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter, *Phys. Rev.* **D73**, 104002 (2006), `gr-qc/0602026`.
39. J. R. van Meter, J. G. Baker, M. Koppitz, and D.-I. Choi, *Phys. Rev.* **D73**, 124011 (2006), `gr-qc/0605030`.
40. M. Campanelli, C. O. Lousto, and Y. Zlochower, *Phys. Rev.* **D74**, 041501 (2006), `gr-qc/0604012`.
41. L. Blanchet, M. S. S. Qusailah, and C. M. Will, *Astrophys. J.* **635**, 508 (2005), `astro-ph/0507692`.
42. T. Damour, and A. Gopakumar, *Phys. Rev.* **D73**, 124006 (2006), `gr-qc/0602117`.
43. J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter, *submitted to Astrophys. J. Lett.* (2006), `astro-ph/0603204`.
44. U. Sperhake (2006), `gr-qc/0606079`.
45. M. Hannam, S. Husa, D. Pollney, B. Brugmann, and N. O’Murchadha (2006), `gr-qc/0606099`.
46. M. Campanelli, C. O. Lousto, and Y. Zlochower (2006), `astro-ph/0608275`.
47. M. A. Scheel, et al. (2006), `gr-qc/0607056`.