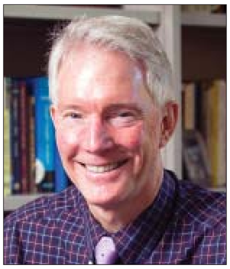

Russell W. Howell

Article

The Matter of Mathematics

Russell W. Howell

This issue of PSCF *is dedicated to mathematics. The general public would likely scoff at the idea that the Christian faith could possibly have any bearing on the subject. Yet for the past thirty-plus years, the Association of Christians in the Mathematical Sciences (ACMS) has devoted much of its energy focusing on precisely that issue. The following lead article begins by asking whether such an effort makes sense, concludes that it does, and highlights several broad categories (with examples) that will hopefully stimulate further conversation. The articles following draw from these categories (or propose new ones) with a special focus on the teaching of mathematics.*

Does faith matter in mathematics? Not according to the Swiss theologian Emil Brunner. In 1937 he suggested a way to view the relationship between various disciplines and the Christian faith. Calling it the "Law of Closeness of Relation," he commented,

> The nearer anything lies to the center of existence where we are concerned with the whole, that is, with man's relation to God and the being of the person, the greater is the disturbance of rational knowledge by sin; the further anything lies from the center, the less the disturbance is felt, and the less difference there is between knowing as a believer or as an unbeliever. This disturbance reaches its maximum in theology and its minimum in the exact sciences and zero in the sphere of the formal. Hence it is meaningless to speak of a "Christian mathematics."[1]

Thus, Brunner holds a nuanced version of the doctrine of noetic depravity: sin affects the reasoning ability of humans, but does so in varying degrees depending on how "close" the object of reasoning is to their relationship with God. Mathematics, being a purely formal discipline, is beyond the reach of any adverse noetic effects. Christians and non-Christians will therefore come to the same mathematical conclusions, so that, for Brunner, the phrase *Christian mathematics* is an oxymoron.

Of course, on one level Brunner is correct. If one agrees to play the game of mathematics, then one implicitly agrees to follow the rules of the game. Different people following these rules will—Christian or not—agree with the conclusions obtained in the same way that different people will agree that, at a particular stage in a game of chess, white can force checkmate in two moves. In this sense mathematical practice is "world-viewishly" neutral. Moreover, the paradigm for mathematical practice has remained relatively unchanged since Euclid published his masterpiece, *The Elements*, in 300 BC. That paradigm is to derive results in the context of an axiomatic system.[2]

It would be a mistake, however, to apply Brunner's dictum to all areas of mathematical inquiry. One can be committed to the mathematical game, but also participate in analyzing it (and even criticizing it) from a metalevel. In doing so, faith perspectives will surely influence the conclusions one comes to on important questions about mathematics.[3] But is the investigation of such questions really a legitimate part of the

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mathematical enterprise? At least two reasons can be given for an affirmative answer: (1) such questions are actually taken up at every annual joint meeting of the American Mathematical Society and the Mathematical Association of America; (2) historically, such questions have always been investigated by the mathematical community. Indeed, David Hilbert, one of the greatest mathematicians of the twentieth century, chose two topics for discussion in conjunction with the oral defense of his doctoral degree. The first related to electromagnetic resistance. The second was to defend an intriguing proposition: "That the objections to Kant's a priori nature of arithmetical judgments are unfounded."[4] Hilbert is credited as being a founder of the school of formalism, which insists that axiomatic procedures in mathematics be followed to the letter. It is thus interesting that even those who held a strict view of mathematical practice and meaning saw the investigation of important metaquestions relating to mathematics as a legitimate undertaking by mathematicians.

Is there a helpful classification for metalevel questions that Christian mathematicians might pursue as they attempt to explore the interaction between their discipline and faith? Arthur Holmes suggests four categories of faith-integration in his well-known book *The Idea of a Christian College*: the foundational, worldview, ethical, and attitudinal.[5] The remainder of this article will look at some developments in mathematics that lead naturally to questions in those categories. It will also suggest (and define) a fifth category for consideration: the *pranalogical*. The ideas presented throughout are by no means meant to be exhaustive, or even representative. It is hoped, though, that they will serve as sufficient triggers for further comment, and for thinking about a wide range of additional metaquestions worthy of investigation.

1. Foundational Issues

Holmes states that curricular studies reveal history and philosophy to be common disciplinary areas considered as foundational in higher education.[6] Within the scope of such an education, each discipline has historical and philosophical components that have shaped its practices, procedures, and paradigms. Mathematics has a particularly rich tradition. This section delineates a sampling of perspectives that lead to important interactions with the Christian faith.

1.1 Logic

Gottlob Frege thought that all of mathematics is reducible to logic. In 1903 he was about to take a big step in pushing through his program. He had just completed his seminal work, *Grundgesetze der Arithmetik* (The Basic Laws of Arithmetic), volume 2. It contained five axioms that, Frege hoped, would lay the necessary groundwork for all of arithmetic. The axioms were supposedly clear logical statements describing universal truths. If this work succeeded, his goal of producing an unshakable logical foundation for mathematics would be realized. Unfortunately, just before the book was to be published, Frege received a disturbing letter from Bertrand Russell, who pointed out that Frege's fifth axiom was in conflict with the other four. In other words, Frege's system was inconsistent. It was too late to stop production, so Frege desperately tried to patch things up and inserted a last-minute appendix in which he modified his fifth axiom. He also openly explained the situation:

> Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion.[7]

It was subsequently shown that Frege's fix did not work, but the effort to ground mathematics on a rock-solid foundation went on. In 1922 the logicians Ernst Zermelo and Abraham Fraenkel produced a collection of axioms that, together with another axiom called the axiom of choice, serves as the basis for a large portion of mathematics (the theory of sets, which can model what one normally thinks of as arithmetic). This axiom set is still in use today, and is referred to as ZFC. Depending on its formulation, ZFC amounts to about ten axioms.[8]

Why use *these* axioms? As we will see in a moment, there is not absolute agreement that ZFC is appropriate, but most mathematicians will give at least two reasons for adopting them: (1) the axioms ring true (i.e., they seem worthy of belief);[9] (2) they produce the desired results. The second condition is important. An axiom set that yields unsatisfactory results is not worth much. But this situation raises an interesting question: what renders these results as "desired"? Is it that they conform to commonly shared empirical experiences, or are they independent ontological entities that mathematicians nevertheless somehow sense? If the former, do different people possibly mean slightly different things when they refer to, say, the number five? If the latter, where are these entities located? In God's mind? Section 1.2, Ontology, briefly explores some of these questions.

Logical Disagreements

One of the disputes regarding the axioms of ZFC arises over the "C" in the acronym, which refers to the axiom of choice. Loosely speaking, this axiom stipulates that, given any collection of non-empty bins, it is possible to select one item from each of them. There is no disagreement among mathematicians over the use of this axiom unless the collection of bins is infinite. Even then, there would not be disagreement if, in a specific instance, there were a specified rule or procedure for the selection. For example, if it were known that the bins consisted of positive integers, one could stipulate—even if some bins had infinitely many positive integers—that the smallest integer is to be chosen from each. If, on the other hand, the only knowledge about the bins were that they contained real numbers (positive or negative), then no constructive procedure could be stipulated ahead of time that would yield a selection. Those accepting the axiom of choice could nevertheless use it to produce a hypothetical selection; those rejecting it would not be able to do so.

Logic and God's Nature

Those who insist that constructive procedures be available in the setting just described likely belong to a school of mathematics known as *Intuitionism*. In general, intuitionists deny that there is any external reality to mathematical objects. Rather, mathematical results are only established by human mental constructions. For them, a mathematical result cannot be established by refuting the claim that the result is false; it must be positively proven within the framework of acceptable intuitionistic assumptions. Thus, intuitionists do not subscribe to the law of excluded middle, which states that, for any proposition P, either P is true or not-P is true.[10] Intuitionists do subscribe to the law of noncontradiction, which states that, for any proposition P, it cannot be the case that P and not-P both hold.[11]

Intuitionism grew out of objections to results that arose in part from the axiom of choice. Its chief proponent was Luitzen Brouwer (1881–1966), who strongly objected to the seeming paradoxes of Georg Cantor's theory of infinite sets. Section 5, Pranalogical Issues, discusses some of these paradoxes. For now we ask a faith-based logical question: Can logical laws be biblically grounded? For example, might 2 Timothy 2:13,[12] "If we are faithless, he remains faithful, for he cannot deny himself," support the law of noncontradiction?[13] What about other laws of logic? The answer to these questions depends on whom you ask.

On the one hand, Sir Michael Dummett (1925–2011), an advocate for intuitionism and a staunch Roman Catholic, rejected classical logic for purely philosophical reasons. He further claimed that his philosophical stance was not influenced in any way by his religious convictions.[14] On the other hand, John Byl, who opts for mathematical realism, attempts to ground a portion of mathematics—including the law of noncontradiction, the axiom of choice, and notions of a completed infinity—on attributes of God found in the scriptures.[15] More generally, Vern Poythress argues that the entire metaphysics of mathematics only holds together coherently because it is part of God's being.[16]

Logic and Gödel

Mathematicians, of course, want coherence, especially in the axioms that help form the building blocks of their edifice. Unfortunately, the theorems that Kurt Gödel produced in 1931 demonstrate that coherence cannot be guaranteed.[17] To explain in full detail the scope of these theorems would go beyond the purpose of this article. Even lengthy treatises by well-known scholars have come under attack by Gödel himself for inaccuracies or misrepresentations.[18] With that caveat out of the way, however, it will be helpful to supply a very brief sketch of Gödel results, as they have important spin-offs for integrative issues. The results apply to any formal axiomatic system that generates an arithmetic capable of addition and multiplication, such as ZFC.[19] In what follows, the phrase *the system* will refer to such an axiomatic system.

Painting with very broad strokes, Gödel created a mechanism for associating a unique number with every well-formed proposition.[20] Thus, if P is a particular proposition of the system it will have a number p associated with it, known as its Gödel number. Gödel then created a proposition G that says, loosely, "The proposition whose Gödel number is g cannot be proved using the results of the system." The remarkable feature about *G* is that its Gödel number actually is *g*. Thus, Gödel found a way to have a self-referential statement without the use of potentially ambiguous indexical terms such as the word *this*. In other words, Gödel created an unambiguous way to formulate a proposition that says, roughly,

G: "This proposition cannot be proved within the system."

Gödel then proved two spectacular results:

Theorem 1: *Within the system, G can be proved if and only if not-G can be proved*.

There are two important implications of Theorem 1:

a. If the system is consistent, then neither *G* nor not-*G* can be proved within it.

b. If the law of excluded middle is allowed, then one of the propositions must be true because they are negations of each other. Thus, if the system is consistent, it contains at least one proposition (either *G* or not-*G*) that is true, but cannot be proved.

Corollary: *If the system is consistent, then G is true.*

This corollary can be made plausible via metareasoning. The proposition *G* says, of itself, that it cannot be proved. But if the system is consistent, then, indeed, *G* cannot be proved, so that *G* asserts the truth (i.e., *G* is true).

Theorem 2: *The system cannot be proved to be consistent using the rules of the system.*

The proof of this theorem proceeds as follows: suppose the system could be proved to be consistent. Then, by the above corollary, we would know that *G* is true, so we would have effectively proven *G*. But then by Theorem 1, we would also have proven not-*G*. Thus, *G* and not-*G* could *both* be proved, which means that the system is not consistent, a contradiction to our assumption. In other words, the assumption that the system can be proved to be consistent leads to an inconsistency. Recall that *the system* refers to any axiomatic system powerful enough to produce an arithmetic capable of addition and multiplication.

Gödel's results have generated a plethora of specious pronouncements. Following is a sample, whose references are not worth reproducing: "Gödel's theorem tells us that nothing can be known for sure"; "Gödel's incompleteness theorem shows that it is not possible to prove that an objective reality exists"; "By equating existence and consciousness, we can apply Gödel's incompleteness theorem to evolution."

Regardless of what these comments actually mean, it is worth noting the apparent common misunderstanding, that Gödel produced *one* theorem. Perhaps an articulation of that misconception is a red flag to consider when evaluating various pontifications.

Are there any lessons that can be legitimately drawn from Gödel's work? Minimally, his results undercut anyone who might subscribe to a "hyper-foundationalist" program, that is, a program that sets out to prove (in a Descartes-like manner) *everything* that is true by starting with a finite set of indisputable truths or axioms. Gödel demonstrated that not even all mathematical truths can be so established with such a program.

Logic and Mechanism

The Oxford philosopher John Lucas has generated much discussion as a result of his claim that Gödel's theorems refute mechanism.[21] Briefly, Lucas points to the corollary of Gödel's Theorem 1, given earlier: *if the system is consistent, then G is true*. Now, Gödel demonstrated that the truth of *G* cannot be established within the formal system that generated it, and any computer (and computer program) is an instantiation of a formal system (presumably, of course, capable of addition and multiplication): it operates according to the rules governed by its hardware-software configuration. Thus, no computer can "know" the truth of *G*. Lucas claims, however, that humans can see that *G* is true.

This is the point at which the argument gets interesting. According to the corollary, humans can see that *G* is true, but only if they know that "the system" is consistent. Yet Gödel's second theorem stipulates that a proof of this consistency is impossible. So, then, how is it that humans can know this fact? Lucas, of course, has responded to this critique. In addition, the well-known physicist Roger Penrose agrees with the main conclusion that Lucas draws about mechanism, though perhaps for slightly different reasons.[22] Most people, however, disagree with the reasoning Lucas employs—even those who agree with his conclusion that mechanism is false.

Logic and God

In September 2013, the scholars Christoph Benzmüller and Bruno Woltzenlogel Paleo drew renewed attention to Gödel's ontological proof of God's existence, which he first gave about ten years after his famous incompleteness theorems.[23] Public interest was also captivated by headlines such as "Computer Scientists 'Prove' God Exists."[24]

Gödel's work is a variation of Anselm's ontological argument, which Anselm introduced in chapter two of his famous *Proslogium*:

> Hence, even the fool is convinced that something exists in the understanding, at least, than which nothing greater can be conceived. For, when he hears of this, he understands it. And whatever is understood, exists in the understanding. And assuredly that, than which nothing greater can be conceived, cannot exist in the understanding alone. For, suppose it exists in the understanding alone: then it can be conceived to exist in reality; which is greater.[25]

Benzmüller and Paleo formulated a version of Gödel's argument into a formal system containing five axioms, three definitions, three theorems, and one corollary. The main conclusion is expressed by Theorem 3: *Necessarily, God exists* (in symbols, $\Box\exists x G(x)$). The axioms can be debated, of course, but the system was verified with the help of mechanical theorem provers.[26]

Logic and Computers

Using computerized theorem provers, or using computers in the assistance of a mathematical proof, remains a controversial issue among mathematicians. The controversy came to a head in 1976, when, at a conference in Toronto, Kenneth Appel and Wolfgang Haken announced that they had, with the help of a computer, produced a proof of the "Four Color Theorem." The theorem states that, given any map, it is possible to color it in such a way that no two *adjacent* regions (such as countries or states) have the same color. The term "adjacent" means that the regions in question share a measurable linear distance, and not that they meet only at a point (as do Arizona and Colorado in a map of the United States). The proof involves a branch of mathematics known as graph theory, and it was the computer-assisted bit that caused the stir.

For starters, the program did something that no human could possibly do: it verified the theorem to be true for hundreds of thousands of possible cases. A proof requiring a human to do something like that would at least violate the criterion of surveyability that Ludwig Wittgenstein popularized.[27]

At a press conference, Appel and Haken were asked several questions about the proof:

> Q: How do you know that the computer itself works properly?
>
> A: We've run the program on different machines and gotten the same results.
>
> Q: How do you know that you've considered all the cases?
>
> A: Actually, the other day someone sent us a letter pointing out that we had missed several cases. But we entered those missing cases into the computer program, and it still came out correct.[28]

The first question can actually be broken down into three parts: How do you know the computer hardware behaves as advertised? How do you know the program you created is correct? How do you know the compiler that translates your program into machine language is correct? There are formal methods for verifying the correctness of computer programs, but hardware and compiler verification have been of very limited scope.

The answer given to the second question is a bit disconcerting, but the two original questions give rise to interesting additional queries: Is there a Christian perspective on the role of computers and mathematical proof? Would such a perspective involve giving up a certain standard of certainty, a standard normally associated with traditional (and surveyable!) proofs?

1.2 Ontology

Many people have an intuition that mathematical truths are independent of humans. In the words of Martin Gardner,

> If two dinosaurs met two others in a forest clearing, there would have been four dinosaurs there—even though the beasts were too stupid to count and there were no humans around to watch.[29]

Additionally, mathematical results seem to remain constant across cultures. The mathematical historian Glen Van Brummelen comments that even pre-modern China, which, for all practical purposes, was mathematically isolated from the rest of the world, exhibits an impressive array of results shared by other cultures, such as the binomial theorem, the solution of polynomial equations via Horner's method, and Gaussian elimination for the solution of systems of linear equations.[30]

Ontological Options

What accounts for this intuition, an intuition that is seemingly reinforced by the apparent similarity of shared conclusions? A common belief is that mathematical objects have some type of objectively real status that we can access in some way. An alternate approach is to suggest that our common brain structure generates both the intuition and shared conclusions.

Supporters for both views can be found among thinkers from within and outside the Christian tradition. The physicist Sir Roger Penrose posits the existence of three separate worlds with complex interactions: the physical world, the mental world, and the (Platonic) mathematical world.[31] His proposal has generated a series of objections and responses.[32] Likewise, the mathematician Alain Connes, who argues for an objective, independent existence of mathematical objects, has debated the biologist Jean-Pierre Changeux, who argues that mathematics is merely a product of neural interactions in the human brain.[33] Problems arise in defending each of these positions. The one reducing mathematics to neural brain interactions has to account for the commonsense notion depicted by the intuition of Gardner, mentioned above. For people with views similar to Penrose and Connes, there is the problem of determining where the mathematical world is located, and coming up with a way to explain how humans have access to this world.

Ontological Realism

The earliest Christian perspective supporting an objectively real mathematics that is independent of human thinking is probably due to Augustine, who locates propositions such as "5 + 7 = 12" in God's mind.[34] With such a view, the ontological question relating to the location of mathematical objects dissolves. Further, the means by which we access these ideas can be explained by our having been created in God's image. In other words, it makes sense that God would create humans whose minds reflect, in some very limited sense, his own rationality.

As attractive as it sounds, there are difficulties with Augustine's view that demand sorting out. Mathematical truths seem to be necessarily true. If so, is God's freedom impaired by the requirement that he must conceive these mathematical thoughts? Christopher Menzel has written in detail on issues like this one.[35] An answer to this question, Menzel states, rests on an appeal to God's nature. To say that God necessarily thinks logical thoughts is only to say that God is rational. He cannot refrain from generating them in the same way that he cannot positively commit a sinful act. He cannot do the latter because he is perfectly good. Likewise, being perfectly rational, he cannot do otherwise than conceive all possible well-formed logical thoughts.

That appears to be a nice solution, but some Christians take issue with it. Roy Clouser, for instance, puts God's thoughts on a different plane from that of humans: "Whereas creatures can't break the law of noncontradiction because they're subject to it, God's transcendent being can't break that law because it doesn't apply to God's being at all."[36]

Those who are comfortable with the idea of logic as part of God's nature, however, have a more serious issue to address. It relates to the contradiction identified by Russell that was mentioned in Section 1.1, Logic. Basically, Russell showed that a set being a member of itself is an incoherent notion. But if God knows all mathematical truths, then he presumably can conceive of all possible sets. This conception is tantamount to a set of all sets, which would mean that such a set has itself as a member. Menzel gets around this difficulty by appealing to what philosophers call an impredicative definition, which is a definition that generalizes over a totality to which the entity being defined belongs. The upshot is that if *S* is a collection (i.e., a set) of sets, then the sets in that collection must have been well formed "before" (in a logical sense) they can be aggregated into the set *S*. Thus, there can be no "set of all sets." To account for God's seeming omniscience of logical constructs, Menzel's model has God collecting these logical entities in a hierarchical type-scheme. This model has been formalized in a theory that includes ZFC, and it is provably consistent relative to ZF. Nevertheless, certain difficulties remain,[37] so more work can profitably be done in this area.

Ontological Nominalism

Problems with mathematical realism have led some thinkers to the view that there are no universals or abstract objects.[38] People belonging to this school are dubbed *Nominalists*, coming from the Latin word *nomen*, meaning name. Thus, for Nominalists, mathematical objects have no objectively real status. Sets, numbers, and propositions are simply convenient naming devices humans have devised to describe common experiences or thoughts.

Historically, many important philosophers have held this view, for example, William of Ockham, John Stuart Mill, and George Berkeley, but there is an important issue for the Nominalist to sort out. It is often referred to as the indispensability argument, popularized by Hilary Putnam and Willard Quine.[39] In a nutshell, the argument points out that mathematics is amazingly applicable to the physical world. One might even say that it is indispensable for science. That being the case, there is good reason to believe in the existence of mathematical entities. It is hard to imagine that something nonexistent in reality can nevertheless apply so well to the physical world.

The Nominalist Hartry Field took this point seriously. His response to the indispensability argument is the work *Science without Numbers*. In it he attempts to show that, so far as their applications go, mathematical theories need not refer to objectively real objects. Instead, the theories merely need to be "conservative" in the sense that they must be consistent and satisfy a few other minimal conditions.[40] Field then develops "nominalistic axioms" that he claims are sufficient for doing science. Many mathematicians, when looking at these axioms, are unconvinced by the argument. To them, the theory that Field built up looks like another form of mathematics, and a very abstract form at that.[41]

Ontology and the Continuum Hypothesis

The continuum hypothesis is due to the work of Georg Cantor (1845–1919), who was the first mathematician to formalize the concept of infinity. Acting out of obedience to carry out his understanding of God's will, Cantor developed a theory of transfinite numbers. It was vigorously opposed by well-known mathematicians such as Leopold Kronecker, who, like Brouwer, was an Intuitionist (see Section 1.1, Logic). According to Joseph Dauben,

> Cantor believed that God endowed the transfinite numbers with a reality making them very special. Despite all the opposition and misgivings of mathematicians in Germany and elsewhere, he would never be persuaded that his results could be imperfect. This belief in the absolute and necessary truth of his theory was doubtless an asset, but it also constituted for Cantor an imperative of sorts. He could not allow the likes of Kronecker to beat him down, to quiet him forever. He felt a duty to keep on, in the face of all adversity, to bring the insights he had been given as God's messenger to mathematicians everywhere.[42]

Cantor showed that infinite sets can be of different sizes. Two infinite sets are the same size (technically, cardinality) if there is a one-to-one correspondence between their elements. Thus, the set of natural numbers ($N = \{1, 2, 3, \ldots\}$) has the same size as the set of even natural numbers ($2N = \{2, 4, 6, \ldots\}$) because there is a one-to-one correspondence between the two sets: $n \leftrightarrow 2n$.

From that standpoint, it seems at face value that all infinite sets would be of the same size, but Cantor showed otherwise. Remarkably, the set A of all real numbers between zero and one cannot be put into a one-to-one correspondence with N. Mathematicians use the symbol "aleph-null" ($\aleph_0$) to designate the cardinality of N, and c (for "continuum") to designate the cardinality of A.

The *continuum hypothesis* (CH) is the assertion that there is no set whose cardinality is between $\aleph_0$ and c. Cantor spent a great deal of effort trying to show that CH is true. At one point, he thought that he had a proof, but he found an error in it. At another point, he thought he had a proof that the hypothesis was false, but again he found an error. He died without knowing the answer.

In 1940 Kurt Gödel took a big step in proving the CH. He showed that, if ZFC is consistent (ZFC is the axiom set discussed in Section 1.1, Logic), then so is the axiom set ZFC + CH. In 1963 the Stanford logician Paul Cohen (1934–2007) finally put the issue to rest, at least in the context of ZFC. Using a technique known as forcing, he showed that, if ZFC is consistent, then so is ZFC + ¬CH (i.e., ZFC + the negation of CH).[43] Collectively, the results of Gödel and Cohen demonstrate that, if ZFC is consistent, then CH can be neither proved nor disproved within that system.

Thus, the question "Is the continuum hypothesis true or false?" actually has four possible answers depending on one's philosophical outlook: (1) Yes, mathematical objects are objectively real entities, so CH must be either true or false, and I think it is true; (2) Yes, CH is either true or false, and I think it is false; (3) Yes, CH is either true or false, but I have no inkling as to what the true situation is; and (4) No, mathematical objects are not objectively real entities, so there is no universal truth of the matter. Gödel and Cohen collectively have shown that, at least under ZFC, CH is neither true nor false.

The outlook people have on the above question is a good indicator of their ontological viewpoint. In some concluding remarks about CH, textbook author Steven Lay writes,

> Thus the continuum hypothesis is undecidable on the basis of the currently accepted axioms for set theory … It remains to be seen whether new axioms will be found that will enable future mathematicians finally to settle the issue.[44]

The thought that the issue can be "settled" probably reveals the author's realist view of mathematical objects.

2. Worldview Issues

Holmes lists four characteristics that comprise a Christian worldview: (1) holistic and integrational (looking at the "big picture"); (2) exploratory (an endless undertaking because a Christian worldview entails that human finiteness is unlikely to exhaust any subject); (3) pluralistic (because Christians, knowing their fallibility, should welcome a variety of perspectives); and (4) confessional or perspectival (a Christian worldview starts with an admixture of beliefs, attitudes, and values).[45]

Some of the topics discussed in the previous section could well qualify as being worldview issues. In what follows we highlight a sampling of additional aspects of mathematics that relate to a Christian worldview.

Unreasonable Effectiveness?

In 1960 the physicist (and eventual Nobel Laureate) Eugene Wigner published an article that has exerted a considerable amount of influence, especially in the past several years.[46] He saw no satisfactory explanation for the phenomenal success that mathematics seemed to enjoy in the quantum world. Matrix procedures that had been successful with the hydrogen atom were abstracted and applied to the helium atom. Wigner states that there was no warrant for this move because the calculation rules were meaningless in this new context. Yet, the application turned out to be miraculous:

> The miracle occurred … [when] the calculation of the lowest energy level of helium … [agreed] with the experimental data within the accuracy of the observations, which is one part in ten million … Surely, in this case we "got something out" of the equations that we did not put in.[47]

Wigner cites other examples and finally concludes by saying,

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.[48]

Wigner finally received a response from the mathematical community in 1980. The computer scientist Richard Hamming published an article in *The American Mathematical Monthly* in which he gave four "partial explanations" that could account for the success of mathematics: (1) mathematicians craft postulates that conform to things they already have observed, so the implications of those postulates would naturally bear success; (2) mathematicians deliberately select the kind of mathematics that, ahead of time, seems appropriate for a given situation, so the success of mathematics is really no surprise; (3) science (and by implication mathematics) answers comparatively few problems, so there is no big success story here; and (4) evolutionary accounts can explain why human reasoning power is successful.[49]

Hamming concludes by saying that his analysis might account for some of the success of mathematics, but does not fully explain it. Given Wigner's experience with the hydrogen-helium story, he would probably take issue with Hamming's second point in any case.

In 2008 the logician and mathematical historian Ivor Grattan-Guinness gave a more thorough response to Wigner. He pays careful attention to how different philosophical schools might view the status of theories: as mere devices for calculation, for example, some forms of positivism; or as explanatory agents, for example, some forms of Platonism. He then argues that, for the most part, mathematical theories develop in a cultural context, are influenced by other theories already in place, and arise in conjunction with "worldly demands." Referencing Karl Popper, he indicates that there may be an element in science that is guesswork. Sometimes one "hits the bullseye," and that just might have been Wigner's situation in the early stages of quantum theory development.[50] This approach is not necessarily at odds with that of Thomas Kuhn, the proponent of paradigm shifts in science.[51] While Grattan-Guinness is not especially sympathetic with Kuhn's explanation of the structure of scientific revolutions, he does "… accept his advocacy of the Gestalt nature of the change."[52]

Grattan-Guinness may have overstated his case somewhat. For example, no physical phenomena guided the formation of complex analysis—a key tool for Wigner. Nevertheless, his case is a powerful one, and it reinforces the danger of the "you can't explain this" attitude that sometimes accompanies the Wigner discussion. It is somewhat reminiscent of "God of the Gaps" theories. A problem with them for Christian apologetics is that, potentially, the gaps that seem to exist with current theories may someday be closed up.

Other attempts to answer Wigner's question from a Christian or theistic perspective are more in line with cosmological "fine-tuning" arguments, some kind of gap/fine-tuning hybrid approach, or an "inference to the best explanation" argument. Mark Steiner agrees with Grattan-Guinness in that he criticizes Wigner for ignoring the failures in science, but nevertheless sees the success of mathematics in science as an argument against naturalism. If guesswork is involved in science, it is interesting that, as a grand strategy, the bullseye so often is hit when the method employed rests on mathematical theories that invariably grew out of human aesthetic criteria. As Brian Green observes, "Physicists … tend to elevate symmetry principles to a place of prominence by putting them squarely on the pedestal of explanation."[53] Steiner sees this outcome as evidence of some sort of privilege that befalls the human species. It makes the universe appear to be "user friendly" and thus of an anthropocentric character. And any form of naturalism, for Steiner, is *ipso facto* nonanthropocentric.[54] The author of this article has produced a fuller elaboration of these aesthetic considerations in the edited volume *C. S. Lewis as Philosopher: Truth, Goodness, and Beauty*.[55]

Aesthetics

What aesthetic principles apply in mathematical theory formation? G. H. Hardy developed several ideas in his book *A Mathematician's Apology*. He states that criteria governing "good" mathematics include economy of expression, depth, unexpectedness, inevitability, and seriousness—qualities that also seem to form standards for good poetry.[56] Two of these standards—inevitability and unexpectedness—seem in conflict: how can something inevitable also be unexpected? In a beautiful mathematical proof, however, there is almost always a clever idea that takes the reader by surprise. The idea often reveals a new insight in a similar way that a brilliant move might reveal an opponent's weakness in a chess match. Then, often with other clever ideas, the proof proceeds to a conclusion that in retrospect is inevitable. A similar line of reasoning might apply to the reading of a beautiful poem. It will contain many phrases or nuances that are delightfully new or unexpected. Yet, at the end—paradoxically—there is a feeling that the prose had to be stated the way it was.

What are some Christian perspectives on mathematical aesthetics? Matt Delong and Kristen Schemmerhorn have produced a short piece,[57] and more work in this area would be welcome.

Chance

In 1998 William Dembski published *The Design Inference*, which is a revision of his PhD dissertation in philosophy for the University of Illinois at Chicago. In it he maps out a mathematical theory for detecting design, and thus can legitimately be considered as a founder of the "Intelligent Design" movement. Essentially, the theory makes use of a "design filter," which operates by asking two questions about phenomena that evidently have no natural law explanations: whether they are statistically very unlikely, and whether they contain independently detectable patterns. If the answer to both questions is yes, then design may be reasonably inferred. Dembski tackles problems, such as determining how unlikely something must be to pass the filter's test, and indicates that the general thrust of his approach conforms with what people do all the time in attributing design to things they encounter.[58]

Dembski's work has generated a considerable amount of controversy—not so much relating to his filter per se, but in his applications of it. An opponent of standard evolutionary explanations for the emergence of life, he is a leading proponent for allowing the teaching of intelligent design as part of the science curriculum in public schools. Along with others, he cites numerous examples of biological systems that purportedly exhibit design as determined by the filter.[59]

Dealing with randomness is awkward for those who view God as sovereign, and also for those who see the universe as a closed, deterministic system. Recently, however, Christian thinkers such as Keith Ward[60] and David Bartholomew[61] have explored the possibility that God may use chance or randomness in fulfilling his purposes for creation. Bartholomew contrasts his thinking with Dembski in the following way:

> The main thesis of the Intelligent Design movement runs counter to the central argument of this book. Here I am arguing that chance in the world should be seen as within the providence of God. That is, chance is a necessary and desirable aspect of natural and social processes which greatly enriches the potentialities of the creation. Many, however, including Sproul, Overman and Dembski, see things in exactly the opposite way. To them, belief in the sovereignty of God requires that God be in total control of every detail and that the presence of chance rules out any possibility of design or of a Designer.[62]

It is not clear that Bartholomew is correct in his description of Dembski's apparent opposition to chance; the main point here, however, is to illustrate two very different approaches to a philosophy of chance that Christian thinkers might take.

The topic of chance has become so important that the Templeton Foundation recently made funding available to help facilitate scholars in their thinking about the issue.[63] James Bradley, the project director for this grant, has listed some interesting examples of randomness that may hint at divine providence.[64] Here are two: (1) The process of diffusion, which involves random molecular motion, delivers nutrients to the approximately ten trillion cells in the human body. Thus, randomness serves a purpose in this instance. (2) Some dynamical systems, for example, Julia sets, produce stable outcomes from random inputs, and other such examples can be found in genetic algorithms and quantum randomness. Thus, order and randomness in these instances are not mutually exclusive.

Bradley has also written about chance for this journal,[65] and for more general readers.[66] Dillard Faries has also published on the topic in this journal.[67] Any additional output that Christian mathematicians might produce in this area will be a welcome contribution to worldview issues.

Culture

Mathematics has had a profound influence on human culture. For example, it can be argued that a significant amount of modern philosophy has been driven by ontological and other problems raised by the practice of mathematics. A portion of a work edited by Howell and Bradley traces this influence from a Christian perspective.[68] Vladimir Tasić has produced a 157-page volume focusing on a single issue: how mathematics has influenced postmodern thought.[69]

Both accounts paint with broad strokes, but the grounds are fertile for Christians expounding on more-targeted influences of mathematics, influences of which the general public might not be aware. Recent articulations of significant issues are not hard to find. To illustrate, Carlos Bovel has argued that a clause in the Westminster Confession can be traced to the geometrical methodological approach in philosophy launched by René Descartes.[70] On a more popular level, a sophisticated mathematical algorithm is the basis for the search engine used by Google. Articles on this topic have appeared in mathematical journals, of course,[71] but accessible books are now on the market, at least for readers with some degree of mathematical sophistication.[72] Information (though outdated) on the process that Google employs is even available for public consumption on Wikipedia.[73]

3. Ethical Issues

The practices by internet companies such as Google have rightly undergone ethical scrutiny by the public. According to Holmes, the values that Christians have will show up—consciously or unconsciously—in their work. In the ethical sphere, an important component for integrating a discipline with the Christian faith involves what ethicists term "middle-level" concepts, which are the mediators between the "facts" uncovered by a discipline and the biblical values of justice and love. This section explores some possibilities for ethical integration in mathematics.

Disciplinary Worth

Christian educators do not all share the same degree of freedom in the profession of their disciplines. The latitude endorsed by their guilds in determining appropriate choice of topics and assigned readings varies considerably. In mathematics, the curricular expectations at the undergraduate level are fairly narrowly focused. Nevertheless, all disciplines share a common concern: whether the discipline itself is worth pursuing.

Two of the standard responses for the worth of mathematics are the *aesthetic argument* (mathematical theories, like great art, have worth simply because of their beauty), and the *future-value argument* (even if a current mathematical theory has no apparent use, theories of mathematics have—historically—eventually resulted in important practical applications). The increasing specialization of mathematics, however, makes these arguments more difficult to sustain. Often, for some highly technical mathematical results, only a dozen or so people fully understand them. If that is the case, the aesthetic and future-value arguments are at least threatened: the value of beautiful things that can be appreciated by only a handful of people can be questioned, and mathematical results must have a certain amount of dissemination if they are to have a reasonable chance of one day finding an application. Michael Veatch has written on this conundrum,[74] and further work from a Christian perspective would be welcome.

Disciplinary Apology

Related to the question of disciplinary worth is the need for Christians to develop an apology for the study of mathematics. The section on aesthetics mentioned an apology by G. H. Hardy. It contains many valuable insights, but was written from a secular perspective and published prior to World War II. Many changes have occurred since then that would no doubt have influenced Hardy's analysis.[75] This author has produced a short apology from a Christian perspective,[76] but a more substantial contribution would render a valuable service to the Christian community.

Disciplinary Pedagogy

The past several years have seen an explosion in pedagogical ideas. In part, it has been driven by the technological revolution. One hears of discussions about MOOCs (massive online open courses), flipped classrooms, IBL (inquiry based learning) practices, and the like. David Klanderman, who specializes in mathematics education, has written on the influence of constructivism in public education,[77] but additional Christian perspectives are needed in evaluating the ever-increasing approaches to education. What, for example, should a Christian response be to pressing factual observations such as the so-called achievement gap in mathematics between various ethnic and social groups? What middle-level concepts can promulgate the biblical values of justice and love in helping overcome the "stereotype threat" that many identifiable groups experience in the mathematical arena?[78]

Should Ethics Influence Mathematics?

Some may claim that ethical considerations should have no bearing on the practice of mathematics. Vern Poythress argues that such a judgment is self-refuting.[79] To see why, label that statement as "C : ethical considerations should have no bearing on the practice of mathematics." Following that as an axiom if you will, it follows that mathematical practice ought not to be influenced by the ethical claim C , which is a self-refuting statement.

4. Attitudinal Issues

Christian mathematicians (indeed, all Christian thinkers) should exhibit practices and affections that grow out of Christian values. According to Holmes,

> If I were teaching symbolic logic, which is as close as a philosopher comes to mathematics, my Christianity would come through with my attitude and integrity far more than in the actual content of the course. A positive, inquiring attitude and a persistent discipline of time and ability express the value I find in learning because of my theology and my Christian commitment.[80]

Holmes goes on to say that these attitudes should affect more than how Christians pursue truth. Their reverence and love for God should also motivate them toward justice (giving all people what they are due, including God), and a desire to act out in practical ways their conviction that every area in the liberal arts—including mathematics—has to do with God.

David Smith gives some nice illustrations of how such attitudes can be played out in teaching the grammar of a foreign language, a subject that is on a similar plane of abstraction as mathematics. He shows how Christian perspectives can be brought to bear in the choice of assigned writing exercises and dialogues used for classroom practice.[81]

Christian mathematics educators can profitably follow Smith's model. Standard exercises in differential equations, for example, can easily be morphed to model phenomena that relate to issues such as ecology or carbon dating that are ripe for Christian involvement. Certain topics by themselves can also serve as springboards for discussion. For example, Wayne Iba has used his training in artificial intelligence to study the proper way in which software programs should render service.[82] What other creative options are possible for Christian mathematicians?

5. Pranalogical Issues

In addition to the four approaches that Holmes delineates, two gospel narratives collectively suggest a fifth category for integrating faith and learning. They share a common feature in that the principles involved are commended by Jesus for their faith.

Pranalogy Defined

The first one, found in Matthew 15:21–28, is the story of the Syrophoenician woman. Her daughter is demon possessed. She begs Jesus for help. In an unusual response, Jesus says, "It is not good to take the children's bread and throw it to the dogs." The woman replies, "Yes, Lord; but even the dogs feed on the crumbs that fall from their masters' table." Jesus then says, "O woman, your faith is great; it shall be done for you as you wish."

The second instance (and the inference that can be drawn from it—see the following paragraph) was highlighted in a chapel address given by Robert Brabenec, in which he referred to an account recorded in Luke 7:1–10.[83] It is the story of a Roman soldier whose servant is desperately ill. In the parallel account given in Matthew 8:5–13, the soldier comes to Jesus and says,

> "Lord, I am not worthy for You to come under my roof, but just say the word, and my servant will be healed. For I also am a man under authority, with soldiers under me; and I say to this one, 'Go!' and he goes, and to another, 'Come!' and he comes, and to my slave, 'Do this!' and he does it." Jesus then says to those around him, "Truly I say to you, I have not found such great faith with anyone in Israel."

Then he heals the servant.

In addition to the praise given by Jesus in these accounts, there is something else that they have in common. The faith of both petitioners came, in part, from their ability to glean a practical spiritual truth by drawing an analogy from what they had learned by experience. The woman did so from behavior she observed among dogs. The soldier likewise understood the implications of having authority by virtue of his occupation, and he applied that knowledge to a trust in the authority that Jesus would have to heal.

This analysis gives rise to an additional category for integrating faith and learning. For lack of a better word, it should probably be called the *pranalogical* because it involves a practical application of an analogy gleaned from one's discipline or life experience. Such an application is the proposed definition of *pranalogy*, a word obtained by combining *pr*actical and *analogy*.

There are several potential pranalogical applications of mathematics that can relate to and even enhance one's Christian faith. Following are some suggestions.

Pranalogical Examples

First, as indicated in Section 1.2, Ontology, Cantor showed that there are actually different sizes of infinity. If the teacher of this theory draws the proper connections, it seems inevitable that, once students see and understand the proof of this result, their notion of God as being infinitely wise, infinitely powerful, or infinitely good, takes on a new and richer meaning, a meaning that would not be possible without seeing that proof.

Of course, other applications involving the infinite are possible. The work of Benoit Mandelbrot and others in developing fractal geometry has led to bizarre sets exhibiting self-similarity and infinite detail.[84] Orbits of points whose starting locations are arbitrarily close together are nevertheless radically different. What pranalogies might Christians meaningfully draw from these ideas?

The second application was brought to light long ago by Bishop George Berkeley. In 1734 he composed an essay entitled "The Analyst; or a Discourse Addressed to an Infidel Mathematician."[85] It is at once a critique of the foundations of calculus and a rebuke of those scientists who deride people of faith for believing in "mysteries," such as the Trinity, that just do not seem to add up. His work closes with a series of 67 pithy rhetorical queries, one of which is "Whether such Mathematicians as cry out against Mysteries, have ever examined their own Principles?"

In other words, Berkeley asserts that, even in mathematics, there are paradoxes. The foundations of calculus have been shored up since Berkeley's time, but paradoxes nevertheless remain. For example, using the axiom of choice, Banach and Tarski were able to show that it is possible to decompose a sphere into only five sections. Then they can be reassembled—without distorting any of the sections in any way—into two completely contiguous spheres of identical size to the first.[86] Surely that is both a mystery and a paradox.[87]

Returning briefly to Cantor's work, the following facts, when put together, are also paradoxical: (1) between any two rational numbers there is an irrational number; (2) between any two irrational numbers there is an irrational number; and (3) these two sets of numbers have no one-to-one correspondence. Thus, there are infinitely more irrational numbers than rational numbers, though infinitely many of both. If pressed to explain this issue, a mathematician might say something like, "Well, that's just how things work when dealing with mysterious concepts like infinity."

Indeed, and if things can get so convoluted in a logically precise, carefully defined system such as mathematics, it should be no surprise when paradoxical ideas arise in the Christian faith. The study of mathematics can thus help cope with these faith paradoxes.

A Pranalogical Caveat

Developing useful pranalogies from one's field of study can be fruitful, but there lurks an obvious danger. In part, it is a danger that accompanies all analogies, but it is especially prominent in mathematics: it is easy to draw analogies that are careless and trite. A well-known mathematician once remarked that the sensitivity of orbits to initial starting locations that Mandelbrot discovered illustrates how God created freedom. Of course, that argument does not hold up. The resulting orbits may be sensitively dependent on their starting locations, and in principle the differences in starting locations may be beyond the capabilities of measurement per the Heisenberg uncertainty principle. Nevertheless, the orbits are still absolutely determined by their starting locations.

Thus, in developing pranalogies one must keep in mind the limits of any model, and in dealing with mysteries ultimately return to Paul's statement in 1 Corinthians 13:12: "For now we see in a glass darkly, but then face to face; now I know in part, but then I will know fully just as I also have been fully known." ◈

Notes

[1]Emil Brunner, *Revelation and Reason* (Philadelphia, PA: Westminster, 1946), 383. Similar statements can be found in other writings by Brunner. See, for example, *The Christian Doctrine of Creation and Redemption* (Cambridge, UK: Clarke, 2002).

[2]A modern axiomatic system has five components: undefined terms (the basic syntactical strings); definitions (composed of undefined terms); axioms (the unquestioned assumptions from which results will be derived); propositions or theorems (the results so obtained); and rules of reasoning (the methods by which axioms and previously proved theorems will be combined to produce new results).

[3]James Bradley, "The Big Questions," in *Mathematics through the Eyes of Faith*, ed. James Bradley and Russell Howell (San Francisco, CA: HarperOne, 2011), 1–14, begins by listing nine such questions.

[4]Constance Reid, *Hilbert* (New York: Springer-Verlag, 1996), 16–17.

[5]Arthur F. Holmes, *The Idea of A Christian College* (Grand Rapids, MI: Eerdmans, 1987). I have altered Holmes's original alphabetical listing to correspond with the order presented in this article.

[6]For Christian colleges, Holmes lists a third area: theology.

[7]Gottlob Frege, *Translations from the Philosophical Writings of Gottlob Frege*, ed. Peter Geach and Max Black (Oxford: Blackwell, 1985), 214.

[8]Technically two of these axioms are actually axiom schemata, each of which contains infinitely many instances. Going into details about this idea, however, is not necessary for the purposes of this article.

[9]The word axiom comes from the Greek *axios* (ἄξιος), meaning worthy.

[10]Note the difference between this statement and the law of bivalence, which says that, for any proposition P, either P is true or P is false.

[11]For further details, see Stewart Shapiro, "Intuitionism: Is Something Wrong with Our Logic?," in *Thinking about Mathematics: The Philosophy of Mathematics* (Oxford: Oxford University Press, 2000), 172–200.

[12]All scripture is from the New American Standard Bible.

[13]This possibility is suggested in James Bradley, Russell Howell, and Kevin Vander Meulen, "Ontology," in *Mathematics through the Eyes of Faith*, ed. Bradley and Howell, 221–44.

[14]Michael Dummett confirmed this fact to me in a private conversation over lunch and tea at Wolfson College, Oxford, in June 2007.

[15]For details, see John Byl, "A Christian View of Mathematics," in *The Divine Challenge: On Matter, Mind, Math and Meaning* (Edinburgh: Banner of Truth, 2004), 254–78.

[16]Vern S. Poythress, "A Biblical View of Mathematics," in *Foundations of Christian Scholarship: Essays in the Van Til Perspective*, ed. Gary North (Vallecito, CA: Ross House, 1976), 158–88.

[17]See Kurt Gödel, "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems," in *Kurt Gödel: Collected Works,* vol. 1, *Publications 1929–1936*, ed. Solomon Feferman et al. (Oxford: Oxford University Press, 1986). It contains Gödel's original 1931 paper in German with an English translation on the facing pages.

[18]In fact, a careful popularized account by two respected academicians was criticized by Gödel himself: that by Ernest Nagel and James Newman, *Gödel's Proof* (New York: New York University Press, 2001).

[19]The model most mathematicians had in mind at the time of Gödel was probably the axiomatic scheme contained in the (eventual) three-volume work, *Principia Mathematica*, by Alfred North Whitehead and Bertrand Russell (Cambridge: Cambridge University Press, 1963).

[20]Very loosely, a "well-formed" proposition is one created by properly obeying the syntactical rules of the axiomatic system being used.

[21]John Lucas, "Minds, Machines, and Gödel," *Philosophy* 36, no. 137 (1961): 112–27, http://users.ox.ac.uk/~jrlucas/mmg.html.

[22]See Sir Roger Penrose, *Shadows of the Mind* (Oxford: Oxford University Press, 1994).

[23]See Kurt Gödel, "Ontological Proof," in *Kurt Gödel: Collected Work*s, vol. 3: *Unpublished Essays and Letters*, ed. Solomon Feferman et al. (Oxford: Oxford University Press, 2001).

[24]See David Knight, "Computer Scientists 'Prove' God Exists," Spiegel Online, abcnews.com, October 27, 2013, http://abcnews.go.com/Technology/computer-scientists-prove-god-exists/story?id=20678984.

[25]Anselm, *Proslogium*, available in a variety of publications and online at http://www.fordham.edu/halsall/basis/anselm-proslogium.asp.

[26]See Christoph Benzmüller and Bruno Woltzenlogel Paleo, "Formalization, Mechanization and Automation of Gödel's Proof of God's Existence," September 10, 2013, http://arxiv.org/pdf/1308.4526v4.pdf.

[27]See Ludwig Wittgenstein, *Remarks on the Foundations of Mathematics* Part III, rev. ed. (Cambridge, MA: MIT, 1983). By *surveyability,* Wittgenstein means that mathematical proofs should be able to be reproduced in the same manner in which an artist can reproduce a picture.

[28]The author of this article was privileged to be at this conference; the reporting of the questions and answers that follow are from memory.

[29]See Martin Gardner, *The Night Is Large: Collected Essays, 1938–1995* (New York: St. Martin's, 1997).

[30]For further elaboration, see Glen Van Brummelen, "Mathematical Truth: A Cultural Study," in *Mathematics in a Postmodern Age: A Christian Perspective*, ed. Russell W. Howell and W. James Bradley (Grand Rapids, MI: Eerdmans, 2001), 45–64.

[31]For details, see Roger Penrose, *The Emperor's New Mind* (Oxford: Oxford University Press, 1989) and Roger Penrose, *Shadows of the Mind* (Oxford: Oxford University Press, 1994).

[32]For example, see Rick Grush and Patricia Churchland, "Gaps in Penrose's Toilings," *Journal of Consciousness Studies* 2, no. 1 (1995): 10–29; Roger Penrose and Stuart Hameroff, "What 'Gaps'?," *Journal of Consciousness Studies* 2, no. 2 (1995): 99–112.

[33]Jean-Pierre Changeux and Alain Connes, *Conversations on Mind, Matter, and Mathematics* (Princeton, NJ: Princeton University Press, 1995).

[34]For a nice elaboration of Augustine's ideas, see James Bradley, "An Augustinian Perspective on the Philosophy of Mathematics," *Journal of the ACMS* (2007), http://www.acmsonline.org/journal/2007/Bradley-Augustinian.htm.

[35]Christopher Menzel, "God and Mathematical Objects," in *Mathematics in a Postmodern Age*, ed. Howell and Bradley, 65–97.

[36]Roy A. Clouser, *The Myth of Religious Neutrality: An Essay on the Hidden Role of Religious Belief in Theories* (South Bend, IN: University of Notre Dame Press, 2005), 229.

[37]See Menzel, "God and Mathematical Objects," in *Mathematics in a Postmodern Age*, ed. Howell and Bradley, footnote 42, 93–94.

[38]Technically, a universal is something that has many instances, such as redness, whereas abstract objects, such as numbers and sets, do not exist in the physical world. This distinction will not be elaborated on any further.

[39]See, for example, Hilary Putnam, "What Is Mathematical Truth?," in *Mathematics, Matter and Method,* vol. 1, *Philosophical Papers*, 2nd ed. (Cambridge: Cambridge University Press, 1979), 60–68; or Willard van Orman Quine, "Things and Their Place in Theories," in *Theories and Things* (Cambridge, MA: Harvard University Press, 1981), 1–23.

[40]See Hartry H. Field, *Science without Numbers: A Defense of Nominalism* (Princeton, NJ: Princeton University Press, 1980).

[41]Anecdotally, when Field's book first came out, many parents of children who feared the subject of mathematics rushed to buy the book because of its title. They hoped it would serve as a "gentler, kinder" introduction to science. They were extremely disappointed, however, when they saw the dense notation!

[42]Joseph Warren Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Princeton, NJ: Princeton University Press, 1979), 291.

[43]Paul Cohen, "The Independence of the Continuum Hypothesis," *Proceedings of the National Academy of Sciences* 50 (1963): 1143–48, http://www.pnas.org/content/50/6/1143.

[44]Steven Lay, *Analysis with an Introduction to Proof,* 5th ed. (Upper Saddle River, NJ: Pearson, 2014), 91.

[45]Holmes, *The Idea of a Christian College*, 58–59.

[46]The Nobel prize was awarded three years after Wigner published his essay. According to the Royal Swedish Academy of Sciences, the basis for Wigner's selection was "his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles."

[47]Eugene Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," *Communications on Pure and Applied Mathematics* 13, no. 1 (1960): 9.

[48]Ibid., 14.

[49]See Richard Wesley Hamming, "The Unreasonable Effectiveness of Mathematics," *American Mathematical Monthly* 87, no. 2 (1980): 81–90.

[50]Ivor Grattan-Guinness, "Solving Wigner's Mystery: The Reasonable (Though Perhaps Limited) Effectiveness of Mathematics in the Natural Sciences," *Springer Science+Business Media* 30, no. 3 (2008): 7–17, http://www.sfu.ca/~rpyke/cafe/reasonable.pdf.

[51]Thomas Kuhn, *The Structure of Scientific Revolutions* (Chicago, IL: The University of Chicago Press, 1996).
[52]Ivor Grattan-Guinness, "The Place of the Notion of Corroboration in Karl Popper's Philosophy of Science," in *Induction and Deduction in the Sciences* (Norwell, MA: Kluwer, 2004), 253.
[53]Brian Green, *The Elegant Universe* (New York: Norton, 1999), 374.
[54]Mark Steiner, *The Applicability of Mathematics as a Philosophical Problem* (Cambridge, MA: Harvard University Press, 1998).
[55]Russell W. Howell, "Lewis's *Miracles* and Mathematical Elegance," in *C. S. Lewis as Philosopher*, ed. Dave Bagit, Jerry Walls, and Gary Habermas (Downers Grove, IL: InterVarsity, 2008), 211–27.
[56]Godfrey Harold Hardy, *A Mathematician's Apology* (Cambridge: Cambridge University Press, 1967) .
[57]For one example, see Matt Delong and Kristen Schemmerhorn, "Beauty," in *Mathematics through the Eyes of Faith*, ed. Bradley and Howell, 143–70.
[58]See William A. Dembski, *The Design Inference: Eliminating Chance through Small Probabilities* (New York: Cambridge University Press, 1988). Dembski also addresses other important issues, such as what it means to have an independently detectable pattern.
[59]See, for instance, Michael J. Behe, *Darwin's Black Box: The Biochemical Challenge to Evolution* (New York: Free Press, 2006).
[60]Keith Ward, *God, Chance and Necessity* (Oxford: Oneworld, 1996).
[61]David J. Bartholomew, *God, Chance and Purpose: Can God Have It Both Ways?* (Cambridge: Cambridge University Press, 2008).
[62]Ibid., 99.
[63]For details, visit Randomness and Divine Providence: Request for Proposals, supported by the John Templeton Foundation, October 1, 2012, http://www.calvin.edu/mathematics/randomnessproject/rfp.html.
[64]See Randomness and Divine Providence: Examples, http://www.calvin.edu/mathematics/randomnessproject/examples.html.
[65]James Bradley, "Randomness and God's Nature," *Perspectives on Science and Christian Faith* 64, no. 2 (2012): 75–89.
[66]See James Bradley, "Chance," in *Mathematics through the Eyes of Faith*, ed. Bradley and Howell, 93–114.
[67]See Dillard W. Faries, "A Personal God, Chance, and Randomness in Quantum Physics," *Perspectives on Science and Christian Faith* 66, no. 1 (2014): 13–22.
[68]See Glen Van Brummelen, "Mathematization in the Premodern Period," in *Mathematics in a Postmodern Age*, ed. Howell and Bradley, 133–222.
[69]Vladimir Tasić, *Mathematics and the Roots of Postmodern Thought* (New York: Oxford University Press, 2001).
[70]Carlos R. Bovell, *By Good and Necessary Consequence: A Preliminary Genealogy of Biblicist Foundationalism* (Eugene, OR: Wipf & Stock, 2009). For an evaluation of this book, see Russell W. Howell, "Inerrancy: A Cartesian *Faux Pas*?," *Books and Culture* (November/December 2014): 8–10.
[71]For example, see Michael Rempe, "Google and the Mathematics of Web Search," in *Eighteenth Biennial Conference Proceedings of the ACMS*, ed. Russell W. Howell (2011): 131–37, http://www.acmsonline.org/conferences/2011/proceedings-2011.pdf.
[72]For example, see Amy N. Langville and Carl D. Meyer, *Google's PageRank and Beyond: The Science of Search Engine Rankings* (Princeton, NJ: Princeton University Press, 2012).
[73]See "PageRank," in *Wikipedia: The Free Encyclopedia*, http://en.wikipedia.org/wiki/PageRank.
[74]See Michael Veatch, "Mathematics and Values," in *Mathematics in a Postmodern Age*, ed. Howell and Bradley, 223–49.
[75]Hardy takes great pride, for example, that no applications will ever be found in his area of research. Ironically, his number-theoretic results are now very useful in modern-day encryption systems.
[76]See Russell Howell, "An Apology," in *Mathematics through the Eyes of Faith*, ed. Bradley and Howell, 245–58.
[77]See David Klanderman, "Teaching and Learning Mathematics: The Influence of Constructivism," in *Mathematics in a Postmodern Age: A Christian Perspective*, ed. Howell and Bradley, 338–59.
[78]For a detailed account of "stereotype threat," see Claude M. Steele, *Whistling Vivaldi: And Other Clues to How Stereotypes Affect Us* (New York: W. W. Norton, 2010).
[79]See Vern S. Poythress, "A Biblical View of Mathematics," in *Foundations of Christian Scholarship: Essays in the Van Til Perspective*, ed. Gary North (Vallecito, CA: Ross House Books, 1976), 158–88. Poythress actually argues for the incoherence of the claim that mathematics should not be influenced by religious belief. I have only slightly adapted his argument for ethics.
[80]Holmes, *The Idea of a Christian College*, 47.
[81]David Smith, "Does God Dwell in the Detail? How Faith Affects (Language) Teaching Processes," in *Engaging the Culture: Christians at Work in Education*, ed. Richard Edlin, Jill Ireland, and Geoff Beech (Blacktown, AU: National Institute for Christian Education, 2014), 131–52.
[82]See Wayne Iba, "Before We Get There, Where Are We Going?," in *Your Virtual Butler: The Making-of (Lecture Notes in Computer Science/Lecture Notes in Artificial Intelligence)*, ed. Robert Trappl (New York: Springer, 2013), 54–69.
[83]The address was given at Wheaton College on March 25, 2009.
[84]For details, see Benoit B. Mandelbrot, *The Fractal Geometry of Nature* (New York: W. H. Freeman, 2002).
[85]George Berkeley, "The Analyst; Or, A Discourse Addressed to an Infidel Mathematician. Wherein it is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith" (1734), ed. David R. Wilkins, available from Kessinger Publishing's Rare Reprints collection and at http://www.maths.tcd.ie/pub/HistMath/People/Berkeley/Analyst/Analyst.html.
[86]For readers of French, see Stefan Banach and Alfred Tarski, "Sur la décomposition des ensembles de points en parties respectivement congruentes," *Fundamenta Mathematicae* 6 (1924): 244–77.
[87]One may object to the claim of paradox on the grounds that the two spheres are each nonmeasurable sets with respect to Lebesgue measure. But that just pushes the conundrum back one step to the question of how there could be such bizarre things as nonmeasurable sets in the first place.